$\mathbf{L^1}$ Full Groups of Flows

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Abstract

We introduce the concept of an L^1 full group associated with a measure-preserving action of a Polish normed group on a standard probability space. These groups carry a natural Polish group topology induced by an L^1 norm. Our construction generalizes L^1 full groups of actions of discrete groups, which have been studied recently by the first author.

We show that under minor assumptions on the actions, topological derived subgroups of L^1 full groups are topologically simple and — when the acting group is locally compact and amenable — are whirly amenable and generically two-generated. L^1 full groups of actions of compactly generated locally compact Polish groups are shown to remember the L^1 orbit equivalence class of the action.

For measure-preserving actions of the real line (also often called measure-preserving flows), the topological derived subgroup of the L^1 full group is shown to coincide with the kernel of the index map, which implies that L^1 full groups of free measure-preserving flows are topologically finitely generated if and only if the flow admits finitely many ergodic components. The latter is in striking contrast to the case of $\mathbb{Z}$-actions, where the number of topological generators is controlled by the entropy of the action. We also prove a reconstruction-type result: the L^1 full group completely characterizes the associated ergodic flow up to flip Kakutani equivalence.

Finally, we study the coarse geometry of L^1 full groups. The L^1 norm on the derived subgroup of the L^1 full group of an aperiodic action of a locally compact amenable group is proved to be maximal in the sense of C. Rosendal. For measure-preserving flows, this holds for the L^1 norm on all of the L^1 full group.

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Chapter 1

Introduction

Full groups were introduced by H. Dye [15] in the framework of measure-preserving actions of countable groups as measurable analogues of unitary groups of von Neumann algebras, by mimicking the fact that the latter are stable under countable cutting and pasting of partial isometries. These Polish groups have since been recognized as important invariants as they encode the induced partition of the space into orbits. A similar viewpoint applies in the setup of minimal homeomorphisms on the Cantor space [23], where likewise the full groups are responsible for the orbit equivalence class of the action.

Full groups are defined to consist of transformations which act by a permutation on each orbit. When the action is free, one can associate with an element h of the full group a cocycle defined by the equation $h(x) = \rho_h(x) \cdot x$. From the point of view of topological dynamics, it is natural to consider the subgroup of those h for which the cocycle map is continuous, which is the defining condition for the so-called topological full groups. The latter has a much tighter control of the action, and encodes minimal homeomorphisms of the Cantor space up to flip-conjugacy (see [23]).

A celebrated result of H. Dye states that all ergodic $\mathbb{Z}$-actions produce the same partition up to isomorphism, and hence the associated full groups are all isomorphic. The first named author has been motivated by the above to seek for the analog of topological full groups in the context of ergodic theory, which was achieved in [40] by imposing integrability conditions on the cocycle. In particular, he introduced L^1 full groups of measure-preserving ergodic transformations, and showed based on the result of R. M. Belinskaja [8] that they also determine the action up to flip-conjugacy. Unlike in the context of Cantor dynamics, these L^1 full groups are uncountable, but they carry a natural Polish topology.

In this work, we widen the concept of an L^1 full group and associate such an object with any measure-preserving Borel action of a Polish normed group (the reader may consult Appendix A for a concise reminder about group norms). Quasi-isometric compatible norms will result in the same L^1 full groups, so actions of Polish boundedly generated groups have canonical L^1 full groups associated with them based on to the work of C. Rosendal [52]. Our study also parallels the generalization of the full group construction introduced by A. Carderi and the first named author in [11], where full groups were defined for Borel measure-preserving actions of Polish groups.

1.1 Main results

Let G be a Polish group with a compatible norm $\|\cdot\|$ and consider a Borel measure-preserving action $G \curvearrowright X$ on a standard probability space (X, μ). The group action defines an orbit equivalence relation $\mathcal{R}_G$ by declaring points $x_1, x_2 \in X$ equivalent whenever $G \cdot x_1 = G \cdot x_2$. The norm induces a metric onto each $\mathcal{R}_G$-class via $D(x_1, x_2) = \inf_{g \in G}\{\|g\| : gx_1 = x_2\}$. Following [11], the **full group** of the action is denoted by $[\mathcal{R}_G]$ and is defined as the collection of all measure-preserving $T \in \mathrm{Aut}(X, \mu)$ that satisfy $x\mathcal{R}_G Tx$ for all $x \in X$. The L^1 **full group** $[G \curvearrowright X]_1$ is given by those $T \in [\mathcal{R}_G]$ for which the map $X \ni x \mapsto D(x, Tx)$ is integrable. This defines a subgroup of $[\mathcal{R}_G]$, and we show in Theorem 2.10 that these groups are Polish in the topology of the norm $\|T\| = \int_X D(x, Tx)\, d\mu(x)$. The strategy of establishing this statement is analogous to that of [12], where the Polish topology for full groups $[\mathcal{R}_G]$ was defined.

Understanding of the structure of various types and variants of full groups often hinges on examining their derived subgroups (also called commutator subgroups). This holds true for our setup as well. Since we're dealing with full groups equipped with non-discrete topologies, we focus on their *topological* derived subgroup, defined as the *closure* of the subgroup generated by commutators.

Theorem 1.1. *The topological derived subgroup of any aperiodic* L^1 *full group is equal to the closed subgroup generated by involutions.*

The argument needed for Theorem 1.1 is quite robust. We extract the idea used in [40], isolate the class of finitely full groups, and show that under mild assumptions on the action, Theorem 1.1 holds for such groups. We provide these arguments in Section 3 and in Corollary 3.16 in particular. Alongside we mention Corollary 3.22 which implies that L^1 full groups of ergodic actions are topologically simple.

For the rest of our results we narrow down the generality of the acting groups, and consider *locally compact* Polish normed groups. In Chapter 4, we show that if $H < G$ is a dense subgroup of a locally compact Polish normed group G then $[H \curvearrowright X]_1$ is dense in $[G \curvearrowright X]_1$. In fact, we prove a considerably stronger statement by showing that for each $T \in [G \curvearrowright X]$ and $\epsilon > 0$ there is $S \in [H \curvearrowright X]$ such that $\operatorname{ess\,sup}_{x \in X} D(Tx, Sx) < \epsilon$.

Recall that a topological group is **amenable** if all of its continuous actions on compact spaces preserve some Radon probability measure, and that it is **whirly amenable** if it is amenable and moreover every invariant Radon measure is supported on the set of fixed points. The following is a combination of Theorem 5.8 and Corollary 5.10.

Theorem 1.2. *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish normed group. Consider the following three statements:*

(1) *G is amenable;*

(2) *the (topological) derived subgroup $\mathfrak{D}([G \curvearrowright X]_1)$ is whirly amenable;*

(3) *the* L^1 *full group $[G \curvearrowright X]_1$ is amenable.*

The implications (1) $\implies$ (2) $\implies$ (3) *always hold. If G is unimodular and the action is free, then the three statements above are all equivalent.*

When the acting group is amenable and the orbits of the action are uncountable, we are able to compute the **topological rank** of the derived L^1 full groups — that is, the minimal number of elements required to generate a dense subgroup of the closure of the commutator subgroup. Theorem 5.19 provides a stronger version of the following.

Theorem 1.3. *Let $G \curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space (X, μ). If all orbits of the action are uncountable, then the topological rank of the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$ is equal to 2.*

It is instructive to contrast the situation with the actions of finitely generated groups, where finiteness of the topological rank of the derived L^1 full group is equivalent to finiteness of the Rokhlin entropy of the action [41].

Our most refined understanding of L^1 full groups is achieved for free actions of $\mathbb{R}$, which are known as **flows**. All the results we described so far are valid for all compatible norms on the acting group. When it comes to the actions of $\mathbb{R}$, however, we consider only the standard Euclidean norm on it. Just like the actions of $\mathbb{Z}$, flows give rise to an important homomorphism, known as the **index map**. Assuming the flow is ergodic, the index map can be described most easily as $[\mathbb{R} \curvearrowright X]_1 \ni T \mapsto \int_X |\rho_T|\, d\mu$, where ρ_T is the cocycle of T. Chapter 6 is devoted to the analysis of the index map for general $\mathbb{R}$-flows.

The most technically challenging result of our work is summarized in Theorem 10.1, which identifies the derived L^1 full group of a flow with the kernel of the index map, and describes the abelianization of $[\mathbb{R} \curvearrowright X]_1$.

Theorem 1.4. *Let $\mathcal{F}$ be a measure-preserving flow on (X, μ). The kernel of the index map is equal to the derived L^1 full group of the flow, and the topological abelianization of $[\mathcal{F}]_1$ is $\mathbb{R}$.*

Theorem 1.4 parallels the known results for $\mathbb{Z}$-actions from [40]. The structure of its proof, however, has an important difference. We rely crucially on the fact that each element of the full group acts in a measure-preserving manner on each orbit. This allows us to use Hopf decomposition (described in Appendix C) in order to separate any given element $T \in [\mathbb{R} \curvearrowright X]_1$ into two parts — recurrent and dissipative. If the acting group were discrete, the recurrent part would reduce to periodic orbits only. This is not at all the case for non-discrete groups, hence we need a new machinery to understand non-periodic recurrent transformations. To cope with this, we introduce the concept of an **intermitted** transformation, which plays the central role in Chapter 8, and which we hope will find other applications.

Theorems 1.3 and 1.4 can be combined to obtain estimates for the topological rank of the whole L^1 full groups of flows, which is the content of Proposition 10.3.

Theorem 1.5. *Let $\mathcal{F}$ be a free measure-preserving flow on a standard probability space (X, μ). The topological rank $\mathrm{rk}([\mathcal{F}]_1)$ is finite if and only if the flow has finitely many ergodic components. Moreover, if $\mathcal{F}$ has exactly n ergodic components then*

$$n + 1 \leq \mathrm{rk}([\mathcal{F}]_1) \leq n + 3.$$

In particular, the topological rank of the L^1 full group of an ergodic flow is equal to either 2, 3 or 4. We conjecture that it is always equal to 2, and more generally that the topological rank of the L^1 full group of any measure-preserving flow is equal to $n + 1$ where n is the number of ergodic components.

Our work connects to the notion of L^1 orbit equivalence, an intermediate notion between orbit equivalence and conjugacy. It goes back to the work of R. M. Belinskaja [8] but recently attracted more attention. Stated in our framework, two flows are L^1 orbit equivalent if they can be conjugated so that the first flow is contained in the L^1 full group of the second and vice versa. A symmetric version of Belinskaja's theorem is that ergodic $\mathbb{Z}$-actions are L^1 orbit equivalent if and only if they are flip conjugate. It is very natural to wonder whether this amazing result has a version for flows. Our Theorem 10.14 implies the following.

Theorem 1.6. *If two measure-preserving ergodic flows are L^1 orbit equivalent, then they admit some cross-sections whose induced transformations[1] are flip-conjugate.*

We do not know whether the above result is optimal, that is, whether having flip-conjugate cross-sections implies L^1 orbit equivalence, but it seems unlikely. It is tempting to think that the correct analogue of Belinskaja's theorem would be a positive answer to the following question.

Question 1.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be free ergodic measure-preserving flows which are L^1 orbit equivalent. Is it true that there is $\alpha \in \mathbb{R}^*$ such that $\mathcal{F}_1$ and $\mathcal{F}_2 \circ m_\alpha$ are isomorphic, where m_α denotes the multiplication by α?*

Let us also mention that Theorem 1.6 implies that there are uncountably many L^1 full groups of ergodic free measure-preserving flows up to (topological) group isomorphism (see Corollary 10.16 and the paragraph right after its proof).

Finally, we also investigate the coarse geometry of the L^1 full groups. We establish that the L^1 norm is maximal (in the sense of C. Rosendal [52], see also Appendix A.2) on the derived subgroup of an L^1 full group of an aperiodic measure-preserving action of

[1] We refer the reader to Definition 10.11 and the paragraph that follows it for details on the measure-preserving transformation one associates to a cross-section.

any locally compact amenable Polish group (Theorem 5.5). For the measure-preserving flows, the L^1 norm is, in fact, maximal on the whole full group (Theorem 10.18).

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1.2 Preliminaries

We assume the reader is familiar with the fundamentals of real analysis and measure theory, as presented in standard textbooks such as [17, 53]. For the necessary results in descriptive set theory, we primarily rely on [31].

1.2.1 Ergodic theory

Our work belongs to the field of ergodic theory, which means that all the constructions are defined and results are proven up to null sets. We occasionally phrase our results as holding "for all x" when strictly speaking they hold only "for almost all x". The only part where certain care needs to be exercised in this regard appears in Chapter 2, where we define L^1 full groups for Borel measure-preserving actions of Polish normed groups. As in [11], these definitions require genuine actions rather than *boolean actions*—a distinction that we clarify at the end of this section. This technicality vanishes when considering the more restrictive setting of measure-preserving actions of locally compact Polish groups.

By a **standard probability space**, we mean a unique (up to isomorphism) separable atomless measure space (X, μ) with $\mu(X) = 1$, i.e., the unit interval $[0, 1]$ equipped with the Lebesgue measure. Occasionally, in Chapter 5 and Appendices D and E, we refer to a **standard Lebesgue space**, by which we mean a separable finite measure space, $\mu(X) < \infty$. Unlike a standard probability space, this concept allows for the presence of atoms and does not require normalization.

Throughout, we frequently work with spaces of measurable functions identified up to sets of measure zero. To simplify notation, we omit explicit references to the underlying measure μ. For example, we write $\mathrm{L}^1(X, \mathbb{R})$ instead of $\mathrm{L}^1(X, \mu, \mathbb{R})$ to denote the Banach space of μ-integrable functions $X \to \mathbb{R}$.

We denote by $\mathrm{Aut}(X, \mu)$ the group of all measure-preserving bijections of (X, μ) up to measure zero. This is a Polish group when equipped with the **weak topology**, defined by $T_n \to T$ if and only if for all $A \subseteq X$ Borel, $\mu(T_n(A) \triangle T(A)) \to 0$. The weak topology is a Polish group topology, see [32, Sec. 1]. Given $T \in \mathrm{Aut}(X, \mu)$, its **support** is the set

$$\operatorname{supp} T = \{x \in X : T(x) \neq x\}.$$

We often refer to measure-preserving bijections as (measure-preserving) **transformations**, although they could more precisely be called *invertible transformations*. Since this work does not involve non-invertible transformations, this terminology should not cause confusion. We also consider (measure-preserving) **partial transformations**, which are Borel bijections $T : A \to B$ between Borel subsets A, B of X satisfying $\mu(T^{-1}(C)) = \mu(C)$ for all Borel $C \subseteq B$. The set A is called the domain of T, denoted $\operatorname{dom} T$, and B is its range, denoted $\operatorname{rng} T$.

A measure-preserving bijection T is called **periodic** when almost all of its orbits are finite. The cardinalities of the finite orbits of T are called the **periods** of T. Periodicity implies the existence of a **fundamental domain** A for T, which is a measurable set that intersects every T-orbit at exactly one point. Since the ambient measure μ is finite, the existence of a fundamental domain actually characterizes periodicity. We recall that a transformation T is called **aperiodic** if it has no periodic points, meaning that all of its orbits are infinite.

When considering actions of full group elements on orbits, we also need to deal with bijections that preserve only the measure class on a possibly infinite σ-finite standard measured space. Such bijections are referred to as **non-singular transformations**.

As explained at the beginning of this section, full groups are constructed for (Borel) **measure-preserving actions** of a given Polish group G on a standard probability space (X, μ). These actions, called *spatial actions*, are Borel maps $\alpha : G \times X \to X$ such that for each $g \in G$, the transformation $\alpha(g)$ is measure-preserving. A related notions is that of *boolean actions*, which are continuous group homomorphisms $G \to \mathrm{Aut}(X, \mu)$. Unlike spatial actions, boolean actions identify transformations up to null sets. Consequently, a boolean action can a priori be lifted to an action map α such that, given $g, h \in G$, $\alpha(gh)x = \alpha(g)\alpha(h)x$ holds merely for *almost* every $x \in X$. As discovered by E. Glasner, B. Tsirelson and B. Weiss, boolean actions (also called near actions) of Polish groups do not admit Borel realizations in general, and even when they do, it could happen that different realizations yield different full groups. This subtlety disappears once we shift our attention to locally compact group actions, which is the case for Chapter 4 and onwards. All boolean actions of locally compact Polish groups admit Borel realizations which are all conjugate up to measure zero (and hence have isomorphic full groups), so null sets can be neglected just as they always are in ergodic theory. We refer the reader to [24, 25] for more information on this topic.

1.2.2 Orbit equivalence relations

Any group action $G \curvearrowright X$ induces the orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$, where two points $x, y \in X$ are $\mathcal{R}_{G \curvearrowright X}$-equivalent whenever $G \cdot x = G \cdot y$. We will usually write this equivalence relation simply as $\mathcal{R}_G$ for brevity. For the actions $\mathbb{Z} \curvearrowright X$ generated by an automorphism $T \in \mathrm{Aut}(X, \mu)$, we denote the corresponding orbit equivalence relation by $\mathcal{R}_T$. For clarity, we may sometimes want to name a measure-preserving action as

α and write $G \overset{\alpha}{\curvearrowright} X$. Then for all $g \in G$ we denote by $\alpha(g)$ the measure-preserving transformation of (X, μ) induced by the action of g.

We encounter various equivalence relations throughout this monograph. An equivalence class of a point $x \in X$ under the relation $\mathcal{R}$ is denoted by $[x]_{\mathcal{R}}$ and the **saturation** of a set $A \subseteq X$ is denoted by $[A]_{\mathcal{R}}$ and is defined to be the union of $\mathcal{R}$-equivalence classes of the elements of A: $[A]_{\mathcal{R}} = \bigcup_{x \in A}[x]_{\mathcal{R}}$. In particular, $[x]_{\mathcal{R}_T}$ is the orbit of x under the action of T. The reader may notice that the notation for a saturation $[A]_{\mathcal{R}}$ resembles that for the full group of an action $[G \curvearrowright X]$ (see Chapter 2). Both notations are standard, and we hope that confusion will not arise, as it applies to objects of different nature — sets and actions, respectively.

1.2.3 Actions of locally compact groups

Consider a measure-preserving action of a locally compact Polish (equivalently, second-countable) group G on a standard Lebesgue space (X, μ). A **complete section** for the action is a measurable set $C \subseteq X$ that intersects almost every orbit, i.e., $\mu(X \setminus G \cdot C) = 0$. A **cross-section** is a complete section $C \subseteq X$ such that for some non-empty neighborhood of the identity $U \subseteq G$ we have $Uc \cap Uc' = \varnothing$ whenever $c, c' \in C$ are distinct. When the need to mention such a neighborhood U explicitly arises, we say that C is a U**-lacunary** cross-section.

With any cross-section C one associates a decomposition of the phase space known as the Voronoi tessellation. Slightly more generally, Appendix E.2 defines the concept of a tessellation over a cross-section, which corresponds to a set $\mathcal{W} \subseteq C \times X$ for which the fibers $\mathcal{W}_c = \{x \in X : (c, x) \in \mathcal{W}\}$, $c \in C$, partition the phase space. Every tessellation $\mathcal{W}$ gives rise to an equivalence relation $\mathcal{R}_{\mathcal{W}}$, where points $x, y \in X$ are deemed equivalent whenever they belong to the same fiber $\mathcal{W}_c$. The projection map $\pi_{\mathcal{W}} : X \to C$ associates with each $x \in X$ the unique $c \in C$ satisfying $x \in \mathcal{W}_c$, and it is therefore defined by the condition $(\pi_{\mathcal{W}}(x), x) \in \mathcal{W}$ for all $x \in X$.

When the action $G \curvearrowright X$ is free, each orbit $G \cdot x$ can be identified with the acting group. Such a correspondence $g \mapsto g \cdot x$ depends on the choice of the anchor point x within the orbit, but suffices to transfer structures invariant under the right translations from the group G onto the orbits of the action. For instance, if the acting group is locally compact, then a right-invariant Haar measure λ can be pushed onto orbits by setting $\lambda_x(A) = \{g \in G : g \cdot x \in A\}$ as discussed in Section 4.2. Freeness of the action $G \curvearrowright X$ gives rise to the **cocycle map** $\rho : \mathcal{R}_{G \curvearrowright X} \to G$, which is well-defined by the condition $\rho(x, y) \cdot x = y$. Elements of the full group $[G \curvearrowright X]$ are characterized as measure-preserving transformations $T \in \mathrm{Aut}(X, \mu)$ such that $(T(x), x) \in \mathcal{R}_{G \curvearrowright X}$ for all $x \in X$. With each $T \in [G \curvearrowright X]$ one may therefore associate the map $\rho_T : X \to G$, also known as the **cocycle map**, and defined by $\rho_T(x) = \rho(x, Tx)$. Both the context and the notation will clarify which cocycle map is being referred to.

1.2.4 Measure-preserving free flows

All the previous concepts appear prominently in the chapters that deal with free measure-preserving *flows*, namely free measure-preserving actions of $\mathbb{R}$ on standard probability spaces. We use the additive notation for such actions: $\mathbb{R} \times X \ni (r, x) \mapsto x + r \in X$. The group $\mathbb{R}$ carries a natural linear order that is invariant under the group operation and can therefore be transferred onto orbits. More specifically, given a free measure-preserving flow $\mathbb{R} \curvearrowright X$, we use the notation $x < y$ whenever x and y belong to the same orbit and $y = x + r$ for some $r > 0$. Every cross-section C of a free flow intersects each orbit in a bi-infinite fashion — each $c \in C$ has a unique successor and a unique predecessor in the order of the orbit. One therefore has a bijection $\sigma_C : C \to C$, called the **first return map** or the **induced map**, which sends $c \in C$ to the next element of the cross-section within the same orbit. We also make use of the gap function that measures the lengths of intervals of the cross-section, i.e., $\mathrm{gap}_C(c) = \rho(c, \sigma_C(c))$, and of the projection function $\pi_C : X \to C$ which takes every $x \in X$ to the largest $c \in C$ such that $c \leq x$.

There is a canonical tessellation associated with a cross-section C which partitions each orbit into intervals between adjacent points of C and is given by

$$\mathcal{W}_C = \{(c, x) \in C \times X : c \leq x < \sigma_C(c)\}.$$

The associated equivalence relation $\mathcal{R}_{\mathcal{W}_C}$ is denoted simply by $\mathcal{R}_C$. It groups points $(x, y) \in \mathcal{R}_{\mathbb{R} \curvearrowright X}$ which belong to the same interval of the tessellation, $\pi_C(x) = \pi_C(y)$. The $\mathcal{R}_C$-equivalence class of $x \in X$ is equal to $[x]_{\mathcal{R}_C} = \pi_C(x) + \big[0, \mathrm{gap}_C(\pi_C(x))\big)$.

Often enough we need to restrict sets and functions to an $\mathcal{R}_C$-class. Since such a need arises very frequently, especially in Chapter 9, we adopt the following shorthand notations. Given a set $A \subseteq X$ and a point $c \in C$, the intersection $A \cap [c]_{\mathcal{R}_C}$ is denoted simply by $A(c)$. Likewise, $\lambda_c^C(A)$ stands for $\lambda(\{t \in \mathbb{R} : c + t \in A \cap [c]_{\mathcal{R}_C}\})$ and corresponds to the Lebesgue measure of the set $A \cap [c]_{\mathcal{R}_C}$.

The phase space X can be identified with the subset $Z_C \subseteq C \times \mathbb{R}$,

$$Z_C = \{(c, t) : 0 \leq t < \mathrm{gap}_C(c)\},$$

via the map $\Psi : Z_C \to X$ given by $\Psi(c, t) = c + t$. Through this identification, λ_c corresponds to the Lebesgue measure on the fiber of Z_C over c. Moreover, uniqueness of the Lebesgue measure implies uniqueness of a finite measure ν on C such that μ is the push-forward by Ψ of the restriction of $\nu \times \lambda$ to Z_C (see, for instance, [37, Prop. 4.3]). The natural disintegration of $(C \times \mathbb{R}, \nu \otimes \lambda)$ corresponds to the disintegration of μ of the form $\mu(A) = \int_C \lambda_c^C(A)\, d\nu(c)$ (see Appendix D for an overview of measure disintegration).

Chapter 2

L^1 full groups of Polish group actions

We begin by introducing the central concept of this work, namely the L^1 full groups of Borel measure-preserving actions of Polish normed groups on a standard probability space. While our primary focus will be on actions of locally compact groups, especially flows, the notion of an L^1 full group can be introduced for actions of arbitrary Polish normed groups. In Section 2.1, we present the definitions in this general setting.

In Section 2.2, we examine the case when there is a natural choice (up to quasi-isometry) of a norm on G, allowing us to speak of *the* L^1 full group of a G-action. This framework is applicable to $G = \mathbb{R}$, yielding the definition of L^1 full groups of measure-preserving flows.

Returning to the general setting, the L^1 full group of an action of a Polish normed group is itself equipped with a natural norm. In Section 2.3, we demonstrate that the resulting metric space is remarkably large: it contains an isometric copy of the infinite-dimensional Banach space $\mathrm{L}^1(X, \mathbb{R})$.

We conclude this chapter in Section 2.4 by establishing the closure of the L^1 full group under the operation of taking induced transformations. This result will play a pivotal role in the subsequent chapters.

2.1 L^1 full groups of Polish normed group actions

A **Polish normed group** is a Polish group equipped with a compatible norm (see Appendix A.1). Let $(G, \|\cdot\|)$ be a Polish normed group, and let $G \curvearrowright X$ be a Borel measure-preserving action on a standard probability space (X, μ). Using the norm, we define a metric D on X, which may take the value $+\infty$, as follows:

$$D(x, y) = \inf_{u \in G} \{ \|u\| : ux = y \} \text{ for } (x, y) \in X \times X. \tag{2.1}$$

Remark 2.1. By definition, the infimum of the empty set is $+\infty$. Thus, $D(x, y) = +\infty$ if and only if x and y belong to distinct G-orbits.

The fact that D is a metric is straightforward except, possibly, for the implication $D(x, y) = 0 \implies x = y$. To justify the latter, let $u_n \in G$, $n \in \mathbb{N}$, be a sequence such that $u_n \to e$ and $u_n x = y$. The elements $u_n^{-1} u_0$, $n \in \mathbb{N}$, belong to the stabilizer of x. By Miller's theorem [46], stabilizers of all points are closed, whence $u_0 = \lim_n u_n^{-1} u_0$ fixes x. Thus, $u_0 x = x$, and $x = y$ as claimed.

Remark 2.2. When the G-orbit of $x_0 \in X$ is identified with the homogeneous space G/H, where H is the stabilizer of x_0, the restriction of the metric D to the G-orbit of

x_0 corresponds to the quotient metric on G/H induced by the right-invariant metric associated with the given norm on G.

We can then use D to define a norm (which may take the value $+\infty$) on $\mathrm{Aut}(X,\mu)$, leading to the definition of L^1 full groups.

Definition 2.3. Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$ on a standard probability space (X,μ), and let $D : X \times X \to \mathbb{R}^{\geq 0} \cup \{+\infty\}$ be the associated metric on X. The L^1**-norm** of an automorphism $T \in \mathrm{Aut}(X,\mu)$ is denoted by $\|T\|_1$ and is defined as the integral

$$\|T\|_1 = \int_X D(x, Tx)\, d\mu(x).$$

In general, many elements of $\mathrm{Aut}(X,\mu)$ may have an infinite norm. The L^1 **full group** of the action consists of those automorphisms for which the norm is finite:

$$[G \curvearrowright X]_1 = \{T \in \mathrm{Aut}(X,\mu) : \|T\|_1 < +\infty\}.$$

Elements of $[G \curvearrowright X]_1$ form a group under composition, as can be readily verified using the triangle inequality for D and the fact that the transformations in the L^1 full group are measure-preserving. Moreover, it is straightforward to check that $\|\cdot\|_1$ defines a norm on $[G \curvearrowright X]_1$. Our primary objective is to prove that this norm $\|\cdot\|_1$ induces a Polish group topology on $[G \curvearrowright X]_1$. Before doing so, however, we establish a connection to the **full group** of the action $G \curvearrowright X$, defined by

$$[G \curvearrowright X] = \left\{T \in \mathrm{Aut}(X,\mu) : (x, T(x)) \in \mathcal{R}_G \text{ for almost all } x \in X\right\}.$$

Note that since the L^1-norm is given by $\|T\|_1 = \int_X D(x,Tx)\, d\mu(x)$, if $\|T\|_1 < +\infty$, then $D(x,Tx) < +\infty$ holds almost surely, implying $T \in [G \curvearrowright X]$. Thus, the L^1 full group $[G \curvearrowright X]_1$ is a subgroup of the full group $[G \curvearrowright X]$.

The full group of the action was shown to be a Polish group by A. Carderi and the first-named author in [11] (where it is referred to as the *orbit* full group). Furthermore, when the fixed group norm $\|\cdot\|$ is bounded, so is D, which implies that $[G \curvearrowright X]_1 = [G \curvearrowright X]$. Consequently, L^1 full groups encompass the full groups studied in [11]. To demonstrate that L^1 full groups are Polish, we will adopt the same approach as in [11]. We first provide an alternative definition of the L^1 full group, from which the Polishness of the topology will follow directly, and then show that the two definitions yield isometrically isomorphic structures. This alternative definition relies on understanding the cocycles associated with elements of the L^1 full group.

Let us introduce some notation from Appendix B. For a standard Borel space Y, we denote by $\mathrm{L}^0(X,Y)$ the space of measurable maps $X \to Y$. When Y is a Polish normed group $(G, \|\cdot\|)$, we define $\mathrm{L}^1(X,G)$ to be the set of all $f \in \mathrm{L}^0(X,G)$ satisfying

$$\int_X \|f(x)\|\, d\mu(x) < +\infty.$$

Furthermore, given a Borel measure-preserving action $G \curvearrowright X$, we define the map $\Phi : \mathrm{L}^0(X, G) \to \mathrm{L}^0(X, X)$ by $\Phi(f)(x) = f(x) \cdot x$ for $f \in \mathrm{L}^0(X, G)$ and $x \in X$.

Definition 2.4. Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish group G on a standard probability space (X, μ). A function $c \in \mathrm{L}^0(X, G)$ is called a **cocycle** of $T \in [G \curvearrowright X]$ if $T(x) = c(x) \cdot x$ holds for almost every $x \in X$.

Note that if we identify $\mathrm{Aut}(X, \mu)$ with a subset of $\mathrm{L}^0(X, X)$, then c is a cocycle of $T \in [G \curvearrowright X]$ precisely when $T = \Phi(c)$.

Definition 2.5. Consider a Borel measure-preserving action $G \curvearrowright X$ of a Polish normed group $(G, \|\cdot\|)$ on a standard probability space (X, μ). The L^1 **pre-full group** PF^1 is defined as

$$\mathrm{PF}^1 = \Phi^{-1}(\mathrm{Aut}(X, \mu)) \cap \mathrm{L}^1(X, G).$$

In other words, the L^1 pre-full group consists of all *integrable* cocycles.

Remark 2.6. When the group norm $\|\cdot\|$ on G is bounded, the integrability condition becomes trivial. In this case, $\mathrm{PF}^1 = \Phi^{-1}(\mathrm{Aut}(X, \mu))$ coincides with the pre-full group PF as defined in [11, p. 91]. The latter was shown to be Polish in the topology of convergence in measure induced by $\mathrm{L}^0(X, G)$, and our next result encompasses this in view of Proposition B.8.

We equip the L^1 pre-full group with the topology induced by $\mathrm{L}^1(X, G)$, which arises from the norm

$$\|f\|_1^{\mathrm{L}^1(X,G)} = \int_X \|f(x)\| \, d\mu(x).$$

By Proposition B.13, $\mathrm{L}^1(X, G)$ is a Polish normed group under pointwise multiplication.

Following the approach in [11, p. 91], we lift the composition law from $\mathrm{Aut}(X, \mu)$ to cocycles as follows: for $f, g \in \mathrm{PF}^1$ and $x \in X$, define the multiplication by

$$(f * g)(x) = f(\Phi(g)(x))g(x),$$

and the inverse[1] by

$$\mathrm{inv}(f)(x) = f(\Phi(f)^{-1}(x))^{-1}.$$

Proposition 2.7. PF^1 *is a Polish group with the multiplication* $(f, g) \mapsto (f * g)$ *and the inverse* $f \mapsto \mathrm{inv}(f)$*. The function* $f \mapsto \|f\|_1^{\mathrm{L}^1(X,G)}$ *is a compatible group norm on* PF^1*, and* $\Phi \restriction_{\mathrm{PF}^1} : \mathrm{PF}^1 \to \mathrm{Aut}(X, \mu)$ *is a continuous homomorphism.*

[1]The symbol f^{-1} is already reserved for the pointwise inverse on $\mathrm{L}^1(X, G)$. To avoid confusion, we introduce a distinct notation for this new operation.

Proof. First of all, we need to show that these operations are well-defined in the sense that the functions $f * g$ and $\mathrm{inv}(f)$ belong to $\mathrm{L}^1(X, G)$ whenever f and g do. To this end, observe that for $f, g \in \mathrm{PF}^1$,

$$\begin{aligned}\|f * g\|_1^{\mathrm{L}^1(X,G)} &= \int_X \|f(\Phi(g)(x))g(x)\| \; d\mu(x) \\ &\leq \int_X \|f(\Phi(g)(x))\| \; d\mu(x) + \int_X \|g(x)\| \; d\mu(x).\end{aligned}$$

Now note that since $\Phi(g)$ is measure-preserving, we have

$$\int_X \|f(\Phi(g)(x))\| \; d\mu(x) = \int_X \|f(x)\| \; d\mu(x),$$

and therefore

$$\|f * g\|_1^{\mathrm{L}^1(X,G)} \leq \int_X \|f(x)\| \; d\mu(x) + \int_X \|g(x)\| \; d\mu(x) = \|f\|_1^{\mathrm{L}^1(X,G)} + \|g\|_1^{\mathrm{L}^1(X,G)}.$$

In particular, $f * g \in \mathrm{L}^1(X, G)$, and thus PF^1 is closed under multiplication. Similarly, $\Phi(f) \in \mathrm{Aut}(X, \mu)$ implies

$$\begin{aligned}\|\mathrm{inv}(f)\|_1^{\mathrm{L}^1(X,G)} &= \int_X \|f(\Phi(f)^{-1}(x))^{-1}\| \, d\mu(x) \\ &= \int_X \|f(x)^{-1}\| \, d\mu(x) = \|f\|_1^{\mathrm{L}^1(X,G)}.\end{aligned}$$

Thus PF^1 is closed under taking inverses. We leave it to the reader to verify that the operation $*$ endows PF^1 with a group structure, where the inverse of an element $f \in \mathrm{PF}^1$ is given by $\mathrm{inv}(f)$. Additionally, we have shown that $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$ defines a group norm on PF^1. It remains to establish the following properties:

- The topology induced by $\mathrm{L}^1(X, G)$ on PF^1 is Polish.
- The topology induced by $\mathrm{L}^1(X, G)$ on PF^1 is a group topology.
- The restriction of the norm $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$ to PF^1 is compatible with this group topology.

Note that the third property follows directly from the first two, as the norm $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$ determines the topology of $\mathrm{L}^1(X, G)$, even though the latter is equipped with distinct group operations.

We are thus left with verifying that the topology induced by $\mathrm{L}^1(X, G)$ on PF^1 is a Polish group topology. To achieve this, we equip X with a Polish topology that induces its standard Borel structure and ensures the continuity of the G-action on X. Such a topology is guaranteed by a well-known result of H. Becker and A. S. Kechris [6, Thm. 5.2.1]. With this topology in place, $\mathrm{L}^0(X, X)$ can be endowed with the topology of convergence in measure, and the evaluation map

$$\Phi : \mathrm{L}^0(X, G) \to \mathrm{L}^0(X, X), \quad \Phi(f)(x) = f(x) \cdot x,$$

becomes continuous by Lemma B.4.

Since the topology of $\mathrm{L}^1(X, G)$ refines that of $\mathrm{L}^0(X, G)$, the restriction of Φ to $\mathrm{L}^1(X, G)$ remains continuous under its Polish group topology. The continuity of the group operations now follows directly from the continuity of Φ, combined with the continuity of the $\mathrm{Aut}(X, \mu)$-action on $\mathrm{L}^1(X, G)$ (Proposition B.11) and the fact that $\mathrm{L}^1(X, G)$ forms a topological group under pointwise multiplication (Proposition B.13).

We now establish that the induced topology on PF^1 is Polish by invoking Alexandrov's theorem. This theorem states that a subspace of a Polish space is Polish under the induced topology if and only if it is a G_δ set (see [31, Thm. 3.9]). The forward direction of Alexandrov's theorem, together with Proposition B.12, implies that the Polish group $\mathrm{Aut}(X, \mu)$ is a G_δ subset of $\mathrm{L}^0(X, X)$. By the continuity of Φ, it follows that $\mathrm{PF}^1 = \Phi^{-1}(\mathrm{Aut}(X, \mu)) \cap \mathrm{L}^1(X, G)$ is a G_δ subset of $\mathrm{L}^1(X, G)$. Consequently, the reverse implication of Alexandrov's theorem ensures that PF^1 is Polish. ∎

Remark 2.8. Our arguments rely fundamentally on the results of H. Becker and A. S. Kechris [6], which allow us to transform Borel actions into continuous ones. We note that the application of [6, Thm. 5.2.1] in the proof of Proposition 2.7 can be replaced with the more straightforward result [6, Thm. 2.6.6]: every Borel G-action admits a Borel embedding into a continuous G-action on a compact Polish space. By equipping this space with the push-forward measure, we can proceed with our analysis, as the Borel embedding ensures that the actions are isomorphic up to a null set.

Let $K \trianglelefteq \mathrm{PF}^1$ denote the kernel of $\Phi \restriction_{\mathrm{PF}^1}$, and let $\|\cdot\|_1^{\mathrm{PF}^1/K}$ denote the quotient norm induced by $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$ (see Proposition A.3 regarding the properties of the quotient norm). The factor group $(\mathrm{PF}^1/K, \|\cdot\|_1^{\mathrm{PF}^1/K})$ is evidently a Polish normed group, and it turns out to be isometrically isomorphic to the L^1 full group introduced in Definition 2.3, as we will now see. Let $\tilde{\Phi} : \mathrm{PF}^1/K \to \mathrm{Aut}(X, \mu)$ denote the homomorphism induced by $\Phi \restriction_{\mathrm{PF}^1}$ onto the factor group.

Proposition 2.9. *The homomorphism* $\tilde{\Phi} : \mathrm{PF}^1/K \to \mathrm{Aut}(X, \mu)$ *establishes an isometric isomorphism between* $(\mathrm{PF}^1/K, \|\cdot\|_1^{\mathrm{PF}^1/K})$ *and* $([G \curvearrowright X]_1, \|\cdot\|_1)$.

Proof. We begin by showing that $\|gK\|_1^{\mathrm{PF}/K} = \|\tilde{\Phi}(gK)\|_1$ holds for any $gK \in \mathrm{PF}^1/K$. By the definition of the quotient norm,

$$\|gK\|_1^{\mathrm{PF}^1/\mathrm{K}} = \inf_{k \in K} \int_X \|g(x)k(x)\| \, d\mu(x).$$

For any fixed $k \in K$, we have $g(x)k(x) \cdot x = g(x) \cdot x$, and therefore

$$D(x, g(x) \cdot x) \le \|g(x)k(x)\| \quad \text{for almost every } x \in X.$$

This readily implies the inequality $\|\tilde{\Phi}(gK)\|_1 \leq \|gK\|_1^{\mathrm{PF}^1/K}$. For the other direction, let $\epsilon > 0$ and consider the set

$$\{(x,u) \in X \times G : g(x) \cdot x = u \cdot x \text{ and } \|u\| \leq D(x, g(x) \cdot x) + \epsilon\}.$$

Using the Jankov–von Neumann uniformization theorem, one may pick a measurable map $g_0 : X \to G$ that satisfies $g_0(x) \cdot x = g(x) \cdot x$ and $\|g_0(x)\| \leq D(x, g(x) \cdot x) + \epsilon$ for almost all $x \in X$. Since $x \mapsto g(x)^{-1} g_0(x) \in K$, we have

$$\begin{aligned} \|\tilde{\Phi}(gK)\|_1 &= \int_X D(x, g(x) \cdot x)\, d\mu(x) \\ &\geq \int_X \left\|g(x) g(x)^{-1} g_0(x)\right\| \, d\mu(x) - \epsilon \\ &\geq \|gK\|_1^{\mathrm{PF}^1/K} - \epsilon. \end{aligned}$$

As ϵ is an arbitrary positive real, we conclude that $\|gK\|_1^{\mathrm{PF}^1/K} = \|\tilde{\Phi}(gK)\|_1$.

It remains to check that $\tilde{\Phi}$ is surjective. For an automorphism $T \in [G \curvearrowright X]_1$, consider the set

$$\{(x,u) \in X \times G : Tx = u \cdot x \text{ and } \|u\| \leq D(x, Tx) + 1\}.$$

Applying the Jankov–von Neumann uniformization theorem once again, we get a map $g \in \mathrm{L}^0(X, G)$ such that $\Phi(g) = T$ and $\|g(x)\| \leq D(x, Tx) + 1$. The latter inequality, together with the assumption that $T \in [G \curvearrowright X]_1$, easily implies that $g \in \mathrm{L}^1(X, G)$, and thus $gK \in \mathrm{PF}^1/K$ is the preimage of T under $\tilde{\Phi}$. ∎

Results discussed thus far can be summarized as follows.

Theorem 2.10. *Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$ on a standard probability space. The L^1 full group $[G \curvearrowright X]_1$ is a Polish normed group relative to the norm $\|T\|_1 = \int_X D(x, Tx)\, d\mu(x)$.*

Remark 2.11. When the acting group is finitely generated and equipped with the word length metric with respect to a finite generating set, it can be shown that the left-invariant metric induced by the norm on the L^1 full group is complete (see [40, Prop. 3.4 and 3.5] and the remark thereafter for a more general statement). Nevertheless, L^1 full groups generally do not admit compatible complete left-invariant metrics, i.e., they are not necessarily CLI groups. For instance, if $G = \mathbb{R}$ is acting by rotation on the circle, the L^1 full group of the action is all of $\mathrm{Aut}(\mathbb{S}^1, \lambda)$, which is not CLI.

Let us point out a possibility to generalize our framework. Given a standard probability space (X, μ), consider an extended Borel metric D on X, i.e., a Borel metric that is allowed to take the value $+\infty$ (Eq. (2.1) provides such an example). Note that the relation $D(x, y) < +\infty$ is an equivalence relation. One can now define the L^1 full group of D in complete analogy with Definition 2.3 as the group of all $T \in \mathrm{Aut}(X,\mu)$ whose norm $\|T\|_D = \int_X D(x, T(x))\, d\mu(x)$ is finite.

Question 2.12. *Suppose that D restricts to a complete separable metric on each equivalence class $\{y \in X : D(x, y) < +\infty\}$, $x \in X$. Is the L^1 full group of D Polish in the topology of the norm $\|\cdot\|_D$?*

2.2 L^1 full groups and quasi-metric structures

When viewed as a normed group, the L^1 full group $[G \curvearrowright X]_1$ depends on the choice of a compatible norm on G. The topological structure on $[G \curvearrowright X]_1$, however, depends only on the quasi-metric structure of the acting group. Recall that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a Polish group G are **quasi-isometric** if there exists a constant $C > 0$ such that for all $g \in G$,

$$\frac{1}{C}\|g\| - C \leq \|g\|' \leq C\|g\| + C.$$

Lemma 2.13. *Let $\|\cdot\|$ and $\|\cdot\|'$ be two quasi-isometric compatible norms on a Polish group G, and let $G \curvearrowright (X, \mu)$ be a Borel measure-preserving action on a standard probability space. The L^1 full groups associated with the two norms are equal as topological groups.*

Proof. The quasi-isometry condition implies that a function $f : X \to G$ satisfies $\int_X \|f(x)\|\, d\mu(x) < +\infty$ if and only if $\int_X \|f(x)\|'\, d\mu(x) < +\infty$. In particular, the L^1 full groups associated with these norms are equal as abstract groups.

Both topologies make the inclusion of $[G \curvearrowright X]_1$ into $\mathrm{Aut}(X, \mu)$ continuous by Proposition 2.7, and, in particular, the inclusion map is Borel. Since injective images of Borel sets by Borel maps are Borel (see, for example, [31, Thm. 15.1]), it follows that both topologies induce the same Borel structure on $[G \curvearrowright X]_1$, which also coincides with the one induced by the weak topology on $\mathrm{Aut}(X, \mu)$. A standard automatic continuity result (originally due to S. Banach [4, Thm. 4 p. 23]) then yields equality of the two topologies (see also the second paragraph following [6, Lem. 1.2.6]). ∎

When a Polish group G admits a canonical choice of the quasi-metric structure, L^1 full groups $[G \curvearrowright X]_1$ are unambiguously defined as topological groups without the need to choose any particular norm on G. This is the case for *boundedly generated* Polish groups—the class of groups identified and studied by C. Rosendal in his treatise [52]. Appendix A.2 provides a succinct review of the concept of maximal norms on boundedly generated Polish groups.

An example of this situation is given by $G = \mathbb{R}$, where the usual Euclidean norm is maximal in the sense of Definition A.5.

Remark 2.14. We will see in the last chapter that the natural L^1 norm on the L^1 full groups of $\mathbb{R}$-actions is *maximal* so that it defines a quasi-metric structure which is canonically associated with the topological group structure.

2.3 Embedding L^1 isometrically in L^1 full groups

We now present a general result on the geometry of L^1 full groups equipped with the L^1 norm $\|\cdot\|_1$, demonstrating that these groups are quite large.

Given a σ-finite measured space $(X, \mathcal{B}, \lambda)$, let $\mathrm{MAlg}_f(X, \lambda)$ denote the space of all finite-measure subsets $B \in \mathcal{B}$, identified up to measure zero. Endow $\mathrm{MAlg}_f(X, \lambda)$ with the metric $d_\lambda(B_1, B_2) = \lambda(B_1 \triangle B_2)$, where $\triangle$ denotes the symmetric difference.

Proposition 2.15. *Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$. If*

$$[G \curvearrowright X]_1 \neq [G \curvearrowright X],$$

then the metric space $(\mathrm{MAlg}_f(\mathbb{R}, \lambda), d_\lambda)$ embeds isometrically into the L^1 *full group of $G \curvearrowright X$ endowed with its* L^1 *metric, and hence so does* $\mathrm{L}^1(X, \mu, \mathbb{R})$.

Proof. $[G \curvearrowright X]$ is a full group, so any of its elements can be written as a product of three involutions belonging to $[G \curvearrowright X]$ by [55]. By assumption, $[G \curvearrowright X]_1 \neq [G \curvearrowright X]$, so there must be an involution $U \in [G \curvearrowright X]$ which does not belong to $[G \curvearrowright X]_1$. Denote by $\mathcal{B}_U$ the σ-algebra on $\operatorname{supp} U$ consisting of U-invariant sets, endowed with the measure given by $\lambda_U(A) = \|U_A\|_1$, where $U_A(x) = U(x)$ if $x \in A$ and $U_A(x) = x$ otherwise. Since $\operatorname{supp} U = \bigcup_n \{x \in \operatorname{supp} U : D(x, U(x)) \leq n\}$, the measure λ_U is σ-finite. Also, λ_U is non-atomic, because so is μ, and infinite, because $U \notin [G \curvearrowright X]_1$. There is only one σ-finite standard atomless infinite measured space up to isomorphism (namely $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$), so we conclude that $(\mathrm{MAlg}_f(\operatorname{supp} U, \lambda_U), d_{\lambda_U})$ is isometric to $(\mathrm{MAlg}_f(\mathbb{R}, \lambda), d_\lambda)$. Composing this isometry with $A \mapsto U_A$, we get the desired isometric embedding $(\mathrm{MAlg}_f(\mathbb{R}, \lambda), d_\lambda) \to [G \curvearrowright X]_1$.

Finally, we observe that $\mathrm{L}^1(X, \mu, \mathbb{R})$ can be embedded into $\mathrm{MAlg}_f(X \times \mathbb{R}, \mu \otimes \lambda)$ by taking a function f to its epigraph, namely the set of all $(x, y) \in X \times \mathbb{R}$ such that $f(x) \leq y \leq 0$ or $0 \leq y \leq f(x)$. Since there is again only one infinite σ-finite standard atomless measured space and $(X \times \mathbb{R}, \mu \otimes \lambda)$ is such a space, we get an isometric embedding $\mathrm{L}^1(X, \mu, \mathbb{R}) \to \mathrm{MAlg}_f(\mathbb{R}, \lambda)$ as wanted. ∎

Remark 2.16. Full groups of actions of Polish groups are always coarsely bounded. In fact, they are coarsely bounded even as discrete groups[2], which is a result due to

[2]Being coarsely bounded as a discrete group is also called the Bergman property.

M. Droste, W. C. Holland and G. Ulbrich [14] (see also [45, Section I.8] for a more general statement which encompasses the non-ergodic case). In particular, the above result is actually a sharp dichotomy: every L^1 full group of a Polish normed group action is either coarsely bounded, or it contains an isometric copy of $\mathrm{L}^1(X,\mu,\mathbb{R})$.

Remark 2.17. Proposition 2.15 significantly improves [41, Prop. 6.9], since $\mathbb{R}^n$, endowed with the ℓ^1 norm, embeds isometrically into $\mathrm{L}^1(X,\mu,\mathbb{R})$.

2.4 Stability under taking induced transformations

Some of the basic properties of L^1 full groups are discussed—in the wider generality of induction friendly finitely full groups—in Chapter 3. The often-used fundamental fact is the closure of L^1 full groups under taking the induced transformations, which is a generalization of [40, Prop. 3.6]. We formulate this in Proposition 2.18.

Let $T \in \mathrm{Aut}(X,\mu)$ be a measure-preserving transformation. Recall that for a measurable subset $A \subseteq X$, the **induced transformation** T_A is supported on A and is defined to be $T^n(x)$ for $x \in A$ where $n \geq 1$ is the smallest integer such that $T^n(x) \in A$. By the Poincaré recurrence theorem, such a map yields a well-defined measure-preserving transformation.

Proposition 2.18. *Let $G \curvearrowright X$ be a Borel measure-preserving action of a Polish normed group $(G, \|\cdot\|)$. For any element $T \in [G \curvearrowright X]_1$ and any measurable set $A \subseteq X$, the induced transformation T_A belongs to $[G \curvearrowright X]_1$ and moreover $\|T_A\|_1 \leq \|T\|_1$.*

Proof. For $n \geq 1$, let A_n be the set of elements of A whose return time is equal to n; note that $X = \bigsqcup_{n\geq 1} \bigsqcup_{i=0}^{n-1} T^i(A_n)$. Let as before $D : \mathcal{R}_G \to \mathbb{R}^{\geq 0}$ be the metric induced by the group norm $\|\cdot\|$ on the orbits of the action. To estimate the value of $\|T_A\|_1$, observe that

$$\begin{aligned}\|T_A\|_1 &= \int_X D(x, T_A x)\, d\mu(x) = \sum_{n=1}^{\infty} \int_{A_n} D(x, T_A x)\, d\mu(x) \\ &= \sum_{n=1}^{\infty} \int_{A_n} D(x, T^n x)\, d\mu(x).\end{aligned}$$

Using the triangle inequality, we get

$$\begin{aligned}
\|T_A\|_1 &\leq \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{A_n} D(T^i x, T^{i+1}x)\, d\mu(x) \\
&= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{T^i(A_n)} D(x, Tx)\, d(\mu\circ T^{-i})(x) \\
\because\ T \text{ preserves } \mu &= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_{T^i(A_n)} D(x, Tx)\, d\mu(x) = \int_X D(x, Tx)\, d\mu(x) = \|T\|_1 .
\end{aligned}$$

Thus $T_A \in [G \curvearrowright X]_1$ and $\|T_A\|_1 \leq \|T\|_1$ as claimed. ■

Chapter 3

Polish finitely full groups

The primary focus of this work is the study of L^1 full groups of Borel measure-preserving actions of Polish normed groups. However, several results are valid in the more general context of what we call Polish finitely full groups. This generalization encompasses L^1 full groups and provides a unified and extended context for addressing topics such as topological simplicity and maximal norms, building on the results of [40, 41].

Beginning with a Polish finitely full group as defined in Section 3.1, we construct in Section 3.2 a natural closed subgroup, termed the symmetric subgroup, which is analogous to V. Nekrashevych's symmetric and alternating topological full groups [48]. We show that this closed subgroup coincides with the topological derived subgroup under a mild hypothesis, satisfied by L^1 full groups, which we call induction friendliness. Section 3.3 explores closed normal subgroups of the symmetric subgroup: we establish their correspondence to invariant sets—a fact that easily yields topological simplicity when the ambient Polish finitely full group is ergodic. Finally, in Section 3.4, we provide a condition on a *normed* induction friendly Polish finitely full group that guarantees the maximality of its norm's restriction to the symmetric subgroup. Maximality is understood in the sense of C. Rosendal, and Appendix A.2 contains a brief reminder of the relevant notions.

3.1 Polish full and finitely full groups

H. Dye defined a subgroup $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ as being **full** when it is stable under the cutting and pasting of its elements along a *countable partition*: given any partition $(A_n)_n$ of X and any sequence $(g_n)_n$ such that the family $(g_n(A_n))_n$ also partitions X, the element $T \in \mathrm{Aut}(X, \mu)$ obtained as the reunion over $n \in \mathbb{N}$ of the restrictions $g_n \restriction_{A_n}$ belongs to $\mathbb{G}$. In particular, the group $\mathrm{Aut}(X, \mu)$ itself is full.

Given any $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$, the group obtained by cutting and pasting elements of $\mathbb{G}$ along countable partitions is the smallest full subgroup containing $\mathbb{G}$. We denote it by $[\mathbb{G}]$ and call it the **full group generated** by $\mathbb{G}$.

Recall that the uniform topology on $\mathrm{Aut}(X, \mu)$ is the topology induced by the uniform metric d_u defined by

$$d_u(T_1, T_2) = \mu(\{x \in X : T_1 x \neq T_2 x\}).$$

The following can essentially be traced back to H. Dye [15, Lem. 5.4].

Proposition 3.1. *The metric d_u is complete on any full group $\mathbb{G}$, and it is separable if and only if the full group is generated by a countable group.*

Proof. Suppose that $(T_n)_n$ is a Cauchy sequence in the full group $\mathbb{G}$. Taking a subsequence, we may assume that $d_u(T_n, T_{n+1}) < 2^{-n}$ for all n. By the Borel–Cantelli lemma, for almost every $x \in X$ there is some $N \in \mathbb{N}$ such that $T_n x = T_N x$ for all $n \geq N$. Let $Tx = T_N x$ for such $N = N(x)$, and note that T is a measure-preserving bijection[1] and $d_u(T_n, T) \leq 2^{-n+1}$. By construction, T is obtained by cutting and pasting the elements T_n of $\mathbb{G}$ along a countable partition so $T \in \mathbb{G}$, since $\mathbb{G}$ is full.

Suppose $\mathbb{G}$ is separable and let Γ be a countable dense subgroup. The group $[\Gamma]$ is a countably generated full group which is dense in $\mathbb{G}$, so $\mathbb{G} = [\Gamma]$ by completeness. The converse is obtained by noting that if Γ generates $\mathbb{G}$, then one can view $\mathbb{G}$ as the full group of the equivalence relation generated by a realization of the action of Γ on (X, μ), which is d_u-separable by [32, Prop. 3.2]. ■

The L^1 full groups that we are considering are not full in the sense of H. Dye unless the norm on the acting Polish group is bounded, a case which was considered earlier in [11]. They nevertheless satisfy the following weaker property.

Definition 3.2. A group $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ of measure-preserving transformations is **finitely full** if for any partition $X = A_1 \sqcup \cdots \sqcup A_n$ and all elements $g_1, \ldots, g_n \in \mathbb{G}$ such that the sets $g_1 A_1, \ldots, g_n A_n$ also partition X, the transformation $T \in \mathrm{Aut}(X, \mu)$, obtained as the reunion over $i \in \{1, \ldots, n\}$ of the restrictions $g_i \restriction_{A_i}$, belongs to $\mathbb{G}$.

We have the following useful relationship between fullness and finite fullness.

Proposition 3.3. *The d_u-closure of any finitely full group $\mathbb{G}$ is equal to the full group $[\mathbb{G}]$ generated by $\mathbb{G}$. Moreover, every element $T \in [\mathbb{G}]$ is a d_u-limit of elements of $\mathbb{G}$ whose support is contained in the support of T.*

Proof. Since full groups are d_u-closed and using the definition of fullness, it suffices to show that every element $T \in [\mathbb{G}]$ is a limit of elements of $\mathbb{G}$ that belong to the full group generated by T.

Since every $T \in [\mathbb{G}]$ is a product of three involutions in $[T]$[2] [55], it suffices to show that every involution in $[\mathbb{G}]$ is a limit of elements of $\mathbb{G}$ whose support is contained in the support of that involution. Let U be such an involution, let $(A_n)_n$ be a partition of X, and let $(g_n)_n$ in $\mathbb{G}$ be such that $Ux = g_n x$ for all $x \in A_n$. Pick a fundamental domain B for U, i.e., $B \cap U(B) = \emptyset$ and $\mathrm{supp}\, U = B \cup U(B)$. If $B_n = A_n \cap B$, then $Ux = g_n x$ for all $x \in B_n$, and, since U is an involution, $Ux = g_n^{-1} x$ for all $x \in U(B_n)$. Let

$$U_n x = \begin{cases} Ux & \text{if } x \in \bigcup_{m \leq n} (B_m \cup U(B_m)), \\ x & \text{otherwise.} \end{cases}$$

[1] This also follows from the fact due to P. Halmos [27] that $\mathrm{Aut}(X, \mu)$ is d_u-complete.

[2] In fact, we only need the much easier fact that every element is a limit of products of two involutions from its full group, which follows by combining Theorem 3.3 and Sublemma 4.3 from [32].

Clearly $U_n \in \mathbb{G}$, since $\mathbb{G}$ is finitely full. Furthermore, $U_n \xrightarrow{d_u} U$ and $\operatorname{supp} U_n \subseteq \operatorname{supp} U$ by construction, which finishes the proof. ■

Consider a finitely full group $\mathbb{G}$ which is a Borel subset of $\mathrm{Aut}(X, \mu)$ and therefore inherits the structure of a standard Borel space. If $\mathbb{G}$ is Polishable, i.e., if it admits a Polish group topology compatible with the Borel structure, then such topology is necessarily unique and must refine the weak topology inherited from $\mathrm{Aut}(X, \mu)$ (standard automatic continuity results can be found, for instance, in [6, Sec. 1.6]). We refer to such Polishable groups $\mathbb{G}$ endowed with their unique Polish group topology refining the weak topology as **Polish finitely full groups**. In this monograph, our motivating example for introducing this class is of course L^1 full groups.

Remark 3.4. For clarity, we adopt the notation $T_n \xrightarrow{\mathbb{G}} T$ to mean convergence of the sequence (T_n) to T in the Polish topology of $\mathbb{G}$.

For any subgroup $G \leq \mathrm{Aut}(X, \mu)$, there is the smallest finitely full group containing G. Note that if $H \leq \mathrm{Aut}(X, \mu)$ is a finite group, then the finitely full group it generates coincides with the full group it generates. This, in particular, applies to the group generated by a periodic transformation with bounded periods.

Proposition 3.5. *Suppose $\mathbb{G}$ is a Polish finitely full group, and $U \in \mathbb{G}$ is a periodic transformation with bounded periods. The topology induced by $\mathbb{G}$ on the full group of U is equal to the uniform topology.*

Proof. The weak and uniform topologies on $[U]$ coincide because U is periodic. We have already mentioned that the topology of $\mathbb{G}$ refines the weak topology. Since $[U]$ is Polish in the uniform topology, by the automatic continuity result [6, Thm. 1.2.6], the topology induced by $\mathbb{G}$ on the full group of U is refined by the uniform topology. Consequently, the uniform topology and the topology induced from $\mathbb{G}$ onto $[U]$ must coincide. ■

We conclude this preliminary discussion with a definition of aperiodicity, which applies to arbitrary subgroups of $\mathrm{Aut}(X, \mu)$. Such a notion was already worked out by H. Dye [15, Sec. 2] when he introduced type II subgroups. An equivalent version, which suffices for our purposes, is as follows.

Definition 3.6. A subgroup $G \leq \mathrm{Aut}(X, \mu)$ is **aperiodic** if it contains a countable subgroup whose action on (X, μ) has no finite orbits.

Since the weak topology on $\mathrm{Aut}(X, \mu)$ is separable and metrizable, every group $G \leq \mathrm{Aut}(X, \mu)$ contains a countable weakly dense subgroup. Therefore, every aperiodic G contains a countable *weakly dense* subgroup whose action on (X, μ) has no finite orbits. Further discussion of aperiodicity can be found in Appendix F.4.

3.2 Derived subgroup and symmetric subgroup

Recall that the *algebraic* derived subgroup of a group G is the subgroup generated by all commutators. If G is additionally equipped with a group topology, the **topological derived subgroup** is defined as the closure of the algebraic derived subgroup. In this work, we do not consider algebraic derived subgroups and use the term **derived subgroup** exclusively to refer to the *topological* derived subgroup.

Our goal in this section is to determine when the derived subgroup of a Polish finitely full group is topologically generated by involutions—that is, when involutions generate a dense subgroup of the derived subgroup. We begin by noting that aperiodic finitely full groups admit many involutions in the sense of [18, p. 384].

Lemma 3.7. *Let $\mathbb{G}$ be a finitely full aperiodic group. For every measurable nontrivial $A \subseteq X$, there is a nontrivial involution $g \in \mathbb{G}$ whose support is contained in A.*

Proof. By Lemma F.13, there is an involution $T \in [\mathbb{G}]$ whose support is equal to A. By the moreover part of Proposition 3.3, T is the d_u-limit of $g_n \in \mathbb{G}$ supported in A. In particular, one of the g_n's is nontrivial and $g = g_n$ satisfies the statement of the lemma. ■

The first and second items of the following definition constitute analogues of V. Nekrashevych's symmetric and alternating topological full groups [48], respectively. In the setup of L^1 full groups, however, these notions coincide, as we will see shortly.

Definition 3.8. Given a Polish finitely full group $\mathbb{G}$, we let

- $\mathfrak{S}(\mathbb{G})$ be the closed subgroup of $\mathbb{G}$ generated by involutions, which we call the **symmetric subgroup of** $\mathbb{G}$.
- $\mathfrak{A}(\mathbb{G})$ be the closed subgroup of $\mathbb{G}$ generated by 3-cycles, i.e., generated by periodic transformations whose non-trivial orbits have size 3.
- $\mathfrak{D}(\mathbb{G})$ be the closed subgroup generated by commutators, called the **derived subgroup**.

All these groups are closed normal subgroups of $\mathbb{G}$, and $\mathfrak{A}(\mathbb{G}) \leq \mathfrak{S}(\mathbb{G}) \cap \mathfrak{D}(\mathbb{G})$ because every 3-cycle is a commutator of two involutions from its full group.

Proposition 3.9. $\mathfrak{A}(\mathbb{G}) = \mathfrak{S}(\mathbb{G})$ *for any aperiodic finitely full group $\mathbb{G}$.*

Proof. We need to show that every involution is a limit of products of 3-cycles. Let $U \in \mathbb{G}$ be an involution, and let D denote a fundamental domain of U; thus $\operatorname{supp} U = D \sqcup U(D)$. By Lemma F.13, one can find an involution $V \in [\mathbb{G}]$ whose support is equal to D. Since $\mathbb{G}$ is finitely full, we may write D as an increasing union $D = \bigcup_n D_n$, $D_n \subseteq D_{n+1}$, where each D_n is V-invariant, and for every $n \in \mathbb{N}$ the transformation V_n induced by V on D_n belongs to the group $\mathbb{G}$ itself. Let $U_n = U_{D_n \sqcup U(D_n)}$ denote the transformation induced by U onto $D_n \sqcup U(D_n)$ and note that $U_n \to U$ in the uniform topology, and hence also in the topology of $\mathbb{G}$ by Proposition 3.5. Our plan is to use the following permutation identity

$$(12)(34) = (12)(23)(24)(23) = (123)(423), \tag{3.1}$$

where U_n corresponds to $(12)(34)$, V_n to (13), and $U_nV_nU_n$ corresponds to (24). To this end, let C_n be a fundamental domain for V_n, put $W_n = U_{C_n \sqcup U(C_n)}$ (which corresponds to the involution (12)), and, at last, set $S_n = W_nV_nW_n$ (corresponding to $(23) = (12)(13)(12)$). Figure 3.1 illustrates the relations between these sets and transformations.

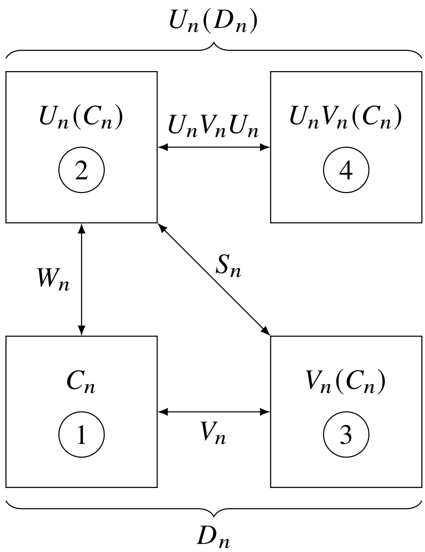

Figure 3.1. The involution U_n is a products of 3-cycles via $(12)(34) = (123)(234)$.

Eq. (3.1) translates into $U_n = \big(W_nS_n\big)\big((U_nV_nU_n)S_n\big)$, so U_n is a product of two 3-cycles, hence it belongs to $\mathfrak{A}(\mathbb{G})$. Since $U_n \xrightarrow{\mathbb{G}} U$, we conclude that $U \in \mathfrak{A}(\mathbb{G})$. ∎

We do not know whether $\mathfrak{A}(\mathbb{G}) = \mathfrak{D}(\mathbb{G})$ holds for all finitely full groups, but here is a convenient sufficient condition.

Definition 3.10. A Polish finitely full group $\mathbb{G}$ is called **induction friendly** if it is stable under taking induced transformations and, furthermore, whenever $T \in \mathbb{G}$ and $(A_n)_n$ is an increasing sequence of T-invariant sets such that $\bigcup_n A_n = A$, then $T_{A_n} \xrightarrow{\mathbb{G}} T_A$.

In the above definition, we require stability under taking the induced transformations, and so T_{A_n} always belongs to $\mathbb{G}$. However, for T-invariant A_n, the assertion $T_{A_n} \in \mathbb{G}$ is already a consequence of $\mathbb{G}$ being finitely full.

Observe that L^1 full groups of measure-preserving actions of Polish normed groups are finitely full and also induction friendly. Indeed, finite fullness follows from a straightforward computation, while induction friendliness is a direct consequence of Proposition 2.18 and the Lebesgue dominated convergence theorem.

Lemma 3.11. *In an induction friendly Polish finitely full group $\mathbb{G}$, every periodic element belongs to $\mathfrak{S}(\mathbb{G})$.*

Proof. Suppose T is periodic. For $n \in \mathbb{N}$, let A_n be the set of $x \in X$ whose T-orbit has cardinality at most n. Each A_n is T-invariant and $\bigcup_n A_n = X$. Moreover, T_{A_n} is periodic, so it can be written as a product of two involutions from its full group (see e.g., [32, Sublem. 4.3]). Since $\mathbb{G}$ is finitely full and the periods of T_{A_n} are bounded, these two involutions belong to $\mathbb{G}$, hence $T_{A_n} \in \mathfrak{S}(\mathbb{G})$. By induction friendliness, $T_{A_n} \xrightarrow{\mathbb{G}} T$, which finishes the proof since $\mathfrak{S}(\mathbb{G})$ is closed in $\mathbb{G}$. ∎

Lemma 3.12. *Let $\mathbb{G}$ be an induction friendly Polish finitely full group. Let $T \in \mathbb{G}$ and $F \subseteq X$ be the aperiodic part of T, i.e.,*

$$F = \{x \in X : T^k x \neq x \text{ for all } k \neq 0\}.$$

For any $A \subseteq X$ such that $F \subseteq \bigcup_{k\in\mathbb{Z}} T^k(A)$ one has $T_A\mathfrak{S}(\mathbb{G}) = T\mathfrak{S}(\mathbb{G})$.

Proof. Since $F \subseteq \bigcup_{k\in\mathbb{Z}} T^k(A)$, the transformation $T^{-1}T_A$ is periodic and therefore belongs to $\mathfrak{S}(\mathbb{G})$ by Lemma 3.11. Hence

$$T\mathfrak{S}(\mathbb{G}) = TT^{-1}T_A\mathfrak{S}(\mathbb{G}) = T_A\mathfrak{S}(\mathbb{G}).$$ ∎

Remark 3.13. The usefulness of the above lemma stems from the following simple observation. If T, T', U, U' satisfy $T\mathfrak{S}(\mathbb{G}) = T'\mathfrak{S}(\mathbb{G})$ and $U\mathfrak{S}(\mathbb{G}) = U'\mathfrak{S}(\mathbb{G})$, then $[T, U] \in \mathfrak{S}(\mathbb{G})$ if and only if $[T', U'] \in \mathfrak{S}(\mathbb{G})$. In particular, for A as in Lemma 3.12, $[T, U] \in \mathfrak{S}(\mathbb{G})$ whenever $[T_A, U] \in \mathfrak{S}(\mathbb{G})$. This fact is used in the proof of the next lemma.

Lemma 3.14. *Suppose $\mathbb{G}$ is an induction friendly Polish finitely full group. If $T, U \in \mathbb{G}$ are aperiodic on their supports, then $[T, U] \in \mathfrak{S}(\mathbb{G})$.*

Proof. Let C be a cross-section for the restriction of $\mathcal{R}_T$ onto $\operatorname{supp} T$. In other words, $C \subseteq X$ is a measurable set satisfying $\bigcup_{i\in\mathbb{Z}} T^i(C) = \operatorname{supp} T$. The induced transformation $U_{X\setminus C}$ commutes with T_C, since their supports are disjoint. We would be done if $\operatorname{supp} U \subseteq \bigcup_{i\in\mathbb{Z}} U^i(X \setminus C)$. Indeed, in this case $T\mathfrak{S}(\mathbb{G}) = T_C\mathfrak{S}(\mathbb{G})$, $U\mathfrak{S}(\mathbb{G}) = U_{X\setminus C}\mathfrak{S}(\mathbb{G})$ by Lemma 3.12 and $[T_C, U_{X\setminus C}]$ is trivial, hence $[T, U] \in \mathfrak{S}(\mathbb{G})$.

Motivated by this observation, we argue as follows. Pick a vanishing nested sequence $(C_n)_{n\in\mathbb{N}}$ of cross-sections for $\mathcal{R}_T \restriction_{\operatorname{supp} T}$, i.e., $C_n \supseteq C_{n+1}$, $\bigcup_{k\in\mathbb{Z}} T^k(C_n) = \operatorname{supp} T$ for all $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} C_n = \varnothing$ (see also Lemma F.11). Such a sequence of cross-sections exists since T is assumed to be aperiodic on its support. Define inductively sets B'_n, $n \in \mathbb{N}$, by setting $B'_0 = X \setminus C_0$, and letting B'_n be the part of $X \setminus C_n$ that does not belong to the U-saturation of any B'_k, $k < n$,

$$B'_n = (X \setminus C_n) \setminus \bigcup_{k<n} \bigcup_{i\in\mathbb{Z}} U^i(B'_k).$$

By construction, saturations under U of the sets B'_n are pairwise disjoint, and the saturation of their union is the whole space, $\bigcup_{i\in\mathbb{Z}} U^i\left(\bigcup_{n\in\mathbb{N}} B'_n\right) = X$, because sets C_n vanish.

Let $B_n = \bigsqcup_{k<n} B'_k$, $B = \bigsqcup_{k\in\mathbb{N}} B'_k$, and note that $U_{B_n}, U_B \in \mathbb{G}$, and $U_{B_k} \xrightarrow{\mathbb{G}} U_B$ by the induction friendliness of $\mathbb{G}$. By construction, the transformations T_{C_n} and U_{B_n} have disjoint supports for each n and, therefore, commute. Since all sets C_n are cross-sections for $\mathcal{R}_T \restriction_{\operatorname{supp} T}$, one has $[T, U_{B_n}] \in \mathfrak{S}(\mathbb{G})$ by Lemma 3.12 and Remark 3.13. Taking the limit as $n \to \infty$, this yields $[T, U_B] \in \mathfrak{S}(\mathbb{G})$. Finally, the U-saturation of B is all of X, and we use Lemma 3.12 and Remark 3.13 once again to conclude that $[T, U] \in \mathfrak{S}(\mathbb{G})$, as claimed. ∎

Proposition 3.15. *If $\mathbb{G}$ is an aperiodic induction friendly Polish finitely full group, then $\mathfrak{S}(\mathbb{G}) = \mathfrak{D}(\mathbb{G})$.*

Proof. The inclusion $\mathfrak{A}(\mathbb{G}) \leq \mathfrak{D}(\mathbb{G})$ holds for any Polish finitely full group, and Proposition 3.9 gives $\mathfrak{S}(\mathbb{G}) \leq \mathfrak{D}(\mathbb{G})$. We therefore concentrate on proving the reverse inclusion: given $T, U \in \mathbb{G}$, we need to check that $[T, U] \in \mathfrak{S}(\mathbb{G})$. Let F_T and F_U be the aperiodic parts of T and U respectively, so that $T\mathfrak{S}(\mathbb{G}) = T_{F_T}\mathfrak{S}(\mathbb{G})$, $U\mathfrak{S}(\mathbb{G}) = U_{F_U}\mathfrak{S}(\mathbb{G})$ by Lemma 3.12. By construction, T_{F_T} and U_{F_U} are aperiodic on their supports and therefore $[T_{F_T}, U_{F_U}] \in \mathfrak{S}(\mathbb{G})$ by Lemma 3.14. It remains to use Remark 3.13 to conclude that necessarily $[T, U] \in \mathfrak{S}(\mathbb{G})$, as needed. ∎

Corollary 3.16. *Let G be a Polish normed group, and let $G \curvearrowright X$ be an aperiodic Borel measure-preserving action on a standard probability space (X, μ). The three subgroups of $[G \curvearrowright X]_1$ introduced in Definition 3.8 coincide:*

$$\mathfrak{D}([G \curvearrowright X]_1) = \mathfrak{A}([G \curvearrowright X]_1) = \mathfrak{S}([G \curvearrowright X]_1).$$

Moreover, they are all equal to the closure of the group generated by periodic elements of $[G \curvearrowright X]_1$.

Proof. The equality $\mathfrak{D}([G \curvearrowright X]_1) = \mathfrak{A}([G \curvearrowright X]_1) = \mathfrak{S}([G \curvearrowright X]_1)$ follows immediately from Propositions 3.9 and 3.15, since $[G \curvearrowright X]_1$ is both finitely full and induction friendly. All these groups are equal to the closure of the group generated by periodic elements of $[G \curvearrowright X]_1$ in view of Lemma 3.11 and the fact that this group obviously contains $\mathfrak{S}([G \curvearrowright X]_1)$. ∎

3.3 Topological simplicity of the symmetric group

We now move on to showing that symmetric subgroups of ergodic Polish finitely full groups are always topologically simple. More generally, we describe the closed normal subgroups of symmetric subgroups of aperiodic Polish finitely full groups. Our argument abstracts from [40, Sec. 3.4]. In particular, we rely on conditional measures associated with subgroups of $\mathrm{Aut}(X, \mu)$, whose construction and basic properties are recalled in Appendix F. We begin with two lemmas on involutions.

Lemma 3.17. *Let $\mathbb{G}$ be an aperiodic Polish finitely full group, let $U, V \in \mathbb{G}$ be two involutions whose supports are disjoint and have the same $\mathbb{G}$-conditional measure. Then U and V are approximately conjugate in $\mathfrak{S}(\mathbb{G})$, i.e., there are $T_n \in \mathfrak{S}(\mathbb{G})$ such that $T_n U T_n^{-1} \xrightarrow{\mathbb{G}} V$.*

Proof. Let A (resp. B) be a fundamental domain of the restriction of U (resp. V) to its support. Then $\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(B)$, and there is an involution $T \in [\mathbb{G}]$ such that $T(A) = B$.

Since $\mathbb{G}$ is finitely full, there is an increasing sequence $(A_n)_n$ of subsets of A such that the involutions T'_n induced by T on $A_n \cup U(A_n)$ belong to $\mathbb{G}$, and $\bigcup_n A_n = A$. Let $B_n = T(A_n) = T'_n(A_n)$ and define involutions $T_n \in \mathbb{G}$ which almost conjugate U to V as follows. For $x \in X$, let

$$T_n x = \begin{cases} Tx & \text{if } x \in A_n \sqcup B_n \\ VTUx & \text{if } x \in U(A_n) \\ UTVx & \text{if } x \in V(B_n) \\ x & \text{otherwise.} \end{cases}$$

For all $n \in \mathbb{N}$ and all $x \in X$, an easy calculation yields that:

- if $x \in (A \cup U(A)) \setminus (A_n \cup U(A_n))$, then $T_n U T_n x = Ux$;
- if $x \in B_n \cup V(B_n)$, then $T_n U T_n x = Vx$;
- and $T_n U T_n x = x$ in all other cases.

In particular, $d_u(T_n U T_n, V) \to 0$ and Proposition 3.5, applied to the full group of the involution UV (which contains both U and V), guarantees that $T_n U T_n \xrightarrow{\mathbb{G}} V$. ∎

Lemma 3.18. *Let $\mathbb{G}$ be an aperiodic Polish finitely full group, let $U \in \mathbb{G}$ be an involution, and let A be a U-invariant subset contained in* $\operatorname{supp} U$. *Suppose that there exists an involution $V \in \mathbb{G}$ such that $V(A)$ is disjoint from* $\operatorname{supp} U$. *Then for all $\mathbb{G}$-invariant functions $f \leq 2\mu_{\mathbb{G}}(A)$, there is an involution $W \in \mathbb{G}$ such that $UWUW$ is an involution whose support has $\mathbb{G}$-conditional measure f.*

Proof. Let $B \subseteq A$ be a fundamental domain for the restriction of U to A and note that $\mu_{\mathbb{G}}(B) = \mu_{\mathbb{G}}(A)/2$. By Maharam's lemma (Theorem F.12), there is $C \subseteq B$ such that $\mu_{\mathbb{G}}(C) = f/4$. The set $D = C \sqcup U(C)$ is U-invariant and satisfies $\mu_{\mathbb{G}}(D) = f/2$. Consider the involution $W \in \mathbb{G}$ defined by

$$Wx = \begin{cases} Vx & \text{if } x \in D \sqcup V(D) \\ x & \text{otherwise.} \end{cases}$$

A straightforward computation shows that $UWUW$ is an involution that coincides with U on D, with VUV on $V(D)$, and is trivial elsewhere. Hence, the support of $UWUW$ is equal to $D \sqcup V(D)$ and has $\mathbb{G}$-conditional measure f. ∎

Given a subgroup $G \leq \mathrm{Aut}(X, \mu)$ and a G-invariant set A, we let G_A stand for the subgroup $\{T \in G : \operatorname{supp} T \subseteq A\}$. Note that G_A is a normal subgroup of G. Our focus is on the case $G = \mathfrak{S}(\mathbb{G})$, where $\mathbb{G}$ is an aperiodic Polish finitely full group. Every subgroup $\mathfrak{S}(\mathbb{G})_A$ is necessarily closed, because the topology of $\mathbb{G}$ refines the weak topology. We show in Theorem 3.20 that all closed normal subgroups of $\mathfrak{S}(\mathbb{G})$ arise in this way.

Proposition 3.19. *Let $\mathbb{G}$ be an aperiodic Polish finitely full group, let $T \in \mathbb{G}$, and let A denote the $\mathbb{G}$-saturation of* $\operatorname{supp} T$. *Then the closed subgroup of $\mathbb{G}$ generated by the $\mathfrak{S}(\mathbb{G})$-conjugates of T contains $\mathfrak{S}(\mathbb{G})_A$.*

Proof. Let the closed subgroup of $\mathbb{G}$ generated by the $\mathfrak{S}(\mathbb{G})$-conjugates of T be denoted by G. By [20, Lem. 7.2], we can find a set $B \subseteq \operatorname{supp} T$ whose T-translates cover $\operatorname{supp} T$ and which satisfies $B \cap T(B) = \varnothing$. Since T-translates of B cover $\operatorname{supp} T$, we conclude that the $\mathbb{G}$-translates of B cover A, and so $\mu_{\mathbb{G}}(B)(x) > 0$ for all $x \in A$. By Maharam's lemma (Theorem F.12), we can find $C \subseteq B$ whose $\mathbb{G}$-conditional measure is everywhere at most $1/4$, and is strictly positive on A. Take $V \in [\mathbb{G}]$ to be an involution such that $V(C \sqcup T(C))$ is disjoint from $C \sqcup T(C)$. Such an involution exists because $\mu_{\mathbb{G}}(C \sqcup T(C)) \leq 1/2$, and so, by Maharam's lemma, we can find a subset contained in $X \setminus (C \sqcup T(C))$ having the same conditional measure as $C \sqcup T(C)$. We can then apply the last item from Proposition F.10.

Let $W \in [\mathbb{G}]$ be an involution such that $\operatorname{supp} W = C$, whose existence is guaranteed by Lemma F.13. Using the facts that $\mathbb{G}$ is finitely full, that $T \in \mathbb{G}$ and that $V, W \in [\mathbb{G}]$, one can find an increasing sequence $(C_n)_n$ of W-invariant subsets of C such that $\bigcup_n C_n = C$ and for each $n \in \mathbb{N}$ both $W_{C_n} \in \mathbb{G}$ and $V_{C_n \sqcup T(C_n) \sqcup V(C_n \sqcup T(C_n))} \in \mathbb{G}$. The transformations $W_{C_n} T W_{C_n} T^{-1}$ belong to G, and are, in fact, involutions whose support is equal to $C_n \sqcup T(C_n)$ and has conditional measure at most $2\mu_{\mathbb{G}}(C) \leq 1/2$. Let us define for brevity

$$\tilde{U}_n = W_{C_n} T W_{C_n} T^{-1} \in G \quad \text{and} \quad \tilde{V}_n = V_{C_n \sqcup T(C_n) \sqcup V(C_n \sqcup T(C_n))} \in \mathbb{G}.$$

For every $n \in \mathbb{N}$, let A_n denote the $\mathbb{G}$-saturation of C_n. Note that $A = \bigcup_n A_n$ and the union is increasing. Every involution supported on A is thus the uniform limit of the involutions it induces on A_n's. By Proposition 3.5, it therefore suffices to show that G contains all the involutions which are supported on some A_n.

Let U be an involution supported on some A_n. Let D be a fundamental domain for the restriction of U to its support. Using Maharam's lemma repeatedly, we can partition D into a countable family $(D_k)_k$ such that

$$\mu_{\mathbb{G}}(D_k) \leq \mu_{\mathbb{G}}(\operatorname{supp}\tilde{U}_n)/2 \quad \text{for all } k \in \mathbb{N}. \tag{3.2}$$

If we let $E_k = D_k \sqcup U(D_k)$, the sequence $(E_k)_k$ forms a partition of $\operatorname{supp} U$ into U-invariant sets. In particular, $U = \lim_k \prod_{i=0}^k U_{E_k}$ in the uniform topology and therefore in the topology of $\mathbb{G}$ as well by Proposition 3.5. Moreover, the support of U_{E_k} has $\mathbb{G}$-conditional measure at most $\mu_{\mathbb{G}}(\operatorname{supp}\tilde{U}_n)$ by Eq. (3.2). The set $\tilde{V}_n(\operatorname{supp}\tilde{U}_n)$ is disjoint from $\operatorname{supp}\tilde{U}_n$ by construction. Lemma 3.18 applies and provides an involution in G whose support has the same conditional measure as that of U_{E_k}. Lemma 3.17 shows that each U_{E_k} belongs to G and therefore also $U \in G$, as needed. ∎

Theorem 3.20. *Let $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ be an aperiodic Polish finitely full group. For any closed normal subgroup $N \leq \mathfrak{S}(\mathbb{G})$, there is a unique $\mathbb{G}$-invariant set A such that $N = \mathfrak{S}(\mathbb{G})_A$.*

Proof. First, observe that for $\mathbb{G}$-invariant A_1 and A_2, any involution $U \in \mathbb{G}$ supported in $A_1 \cup A_2$ decomposes into the product of one involution supported in A_1, and one supported in A_2. It follows that the closed group generated by $\mathfrak{S}(\mathbb{G})_{A_1} \cup \mathfrak{S}(\mathbb{G})_{A_2}$ is equal to $\mathfrak{S}(\mathbb{G})_{A_1 \cup A_2}$. Also, by Proposition 3.5, whenever $(A_n)_n$ is an increasing sequence of $\mathbb{G}$-invariant sets, one has

$$\overline{\bigcup_n \mathfrak{S}(\mathbb{G})_{A_n}} = \mathfrak{S}(\mathbb{G})_{\bigcup_n A_n}.$$

The set $\{A \in \mathrm{MAlg}(X,\mu) : A \text{ is } \mathbb{G}\text{-invariant and } \mathfrak{S}(\mathbb{G})_A \leq N\}$ is thus directed and is closed under the countable unions. It therefore admits a unique maximum element, which is the set A we seek. Indeed, $\mathfrak{S}(\mathbb{G})_A \leq N$, and the reverse inclusion is a direct consequence of Proposition 3.19.

It remains to argue that the set A satisfying $N = \mathfrak{S}(\mathbb{G})_A$ is unique. Suppose towards a contradiction that $\mathfrak{S}(\mathbb{G})_{A_1} = \mathfrak{S}(\mathbb{G})_{A_2}$for $A_1 \neq A_2$. By symmetry, we may assume that $\mu(A_1 \setminus A_2) > 0$. Lemma 3.7 provides an involution $V \in \mathbb{G}$ whose support is nontrivial and is contained in $A_1 \setminus A_2$, thus $V \in \mathfrak{S}(\mathbb{G})_{A_1}$ but $V \notin \mathfrak{S}(\mathbb{G})_{A_2}$, contradicting $\mathfrak{S}(\mathbb{G})_{A_1} = \mathfrak{S}(\mathbb{G})_{A_2}$. ∎

Corollary 3.21. *Let $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ be an aperiodic Polish finitely full group. The group $\mathfrak{S}(\mathbb{G})$ is topologically simple if and only if $\mathbb{G}$ is ergodic.*

Proof. If $\mathbb{G}$ is ergodic, then $\mathfrak{S}(\mathbb{G})$ is topologically simple by Theorem 3.20. Conversely, suppose that $\mathbb{G}$ is not ergodic and let $A \subseteq X$ be a $\mathbb{G}$-invariant set with $\mu(A) \notin \{0,1\}$. Then $\mathfrak{S}(\mathbb{G})_A$ is a normal subgroup of $\mathbb{G}$ which is neither trivial nor equal to $\mathfrak{S}(\mathbb{G})$ as a consequence of Lemma 3.7 applied to A and its complement. ■

Specifying the corollary above to L^1 full groups and using Corollary 3.16, we obtain the following result.

Corollary 3.22. *Let G be a Polish normed group, and let $G \curvearrowright X$ be an aperiodic Borel measure-preserving action on a standard probability space (X,μ). The topological derived subgroup of the L^1 full group of the action is topologically simple if and only if the action is ergodic.*

3.4 Maximal norms on the derived subgroup

The purpose of this section is to establish sufficient conditions for a norm on the derived subgroup of an induction friendly Polish finitely full group to be maximal in the sense of Section 2.2. Our argument follows closely the one given in [41, Sec. 6.2] for amenable graphings. The main application of Proposition 3.25 will be given in Theorem 5.5, but we hope that the setup of this section can be useful in other contexts, such as φ-integrable full groups [10].

Definition 3.23. A norm $\|\cdot\|$ on a subgroup $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ is **additive** if $\|TS\| = \|T\| + \|S\|$ for all $T, S \in \mathbb{G}$ with disjoint supports.

The following lemma parallels [41, Lem. 6.4] and is the key to showing that the norm on the derived subgroup is both coarsely proper and large-scale geodesic.

Lemma 3.24. *Let $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ be a finitely full Polish group, and suppose that $\|\cdot\|$ is a compatible additive norm on $\mathbb{G}$. For any periodic $U \in \mathbb{G}$ with bounded periods and for every $n \in \mathbb{N}$, there are periodic elements $U_1, \dots, U_n \in \mathbb{G}$ such that*

$$U = U_1 \cdots U_n \text{ and } \|U_i\| = \frac{\|U\|}{n} \text{ for every } 1 \leq i \leq n.$$

Proof. Let $M = \|U\|$ and $A \subseteq X$ be a fundamental domain for U. We may identify A with the interval $[0,\mu(A)]$ endowed with the Lebesgue measure. Put $A_t = [0,t] \cap A$, $0 \leq t \leq \mu(A)$, and let $B_t = \bigcup_{n\in\mathbb{Z}} U^n(A_t)$ be the U-saturation of A_t. Note that $U_{B_t} \in \mathbb{G}$ for all $t \in [0,\mu(A)]$ since B_t is U-invariant and $\mathbb{G}$ is finitely full, and that $t \mapsto B_t$ is continuous.

The map $[0,\mu(A)] \ni t \mapsto U_{B_t} \in [U] \subseteq \mathbb{G}$ is thus continuous with respect to the uniform topology on $[U]$, and therefore also with respect to the topology of $\mathbb{G}$ by Proposition 3.5. Whence the function $\psi : [0,\mu(A)] \to \mathbb{R}$ given by $\psi(t) = \left\|U_{B_t}\right\|$ is also continuous.

We have $\psi(0) = 0$ and $\psi(\mu(A)) = M$, so the intermediate value theorem yields existence of reals $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \mu(A)$ such that $\psi(t_i) = \frac{iM}{n}$ for all $i \in \{0, \ldots, n\}$. Set $C_i = B_{t_i} \setminus B_{t_{i-1}}$ for $i \in \{1, \ldots, n\}$. By construction, each C_i is U-invariant and $X = \bigsqcup_{i=1}^n C_i$. Putting $U_i = U_{A_i}$, we get $U = \prod_{i=1}^n U_i$. Finally for each $i \in \{1, \ldots, n\}$ the equality $C_i = B_{t_i} \setminus B_{t_{i-1}}$ and additivity of the norm gives

$$\psi(t_i) = \|U_{B_{t_i}}\| = \|U_i U_{B_{t_{i-1}}}\| = \|U_i\| + \|U_{B_{t_{i-1}}}\| = \|U_i\| + \psi(t_{i-1}),$$

hence $\|U_i\| = \frac{\|U\|}{n}$ for all $i \leq n$, as needed. ∎

Proposition 3.25. *Let $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ be an induction friendly Polish finitely full group and let $\|\cdot\|$ be a compatible additive norm on it. If the set of periodic elements is dense in $\mathfrak{D}(\mathbb{G})$, then $\|\cdot\|$ is a maximal norm on $\mathfrak{D}(\mathbb{G})$.*

Proof. In view of Proposition A.10, it suffices to show that $\|\cdot\|$ is both large-scale geodesic (see Definition A.8) and coarsely proper (see Definition A.9). Note that induction friendliness yields density in $\mathfrak{D}(\mathbb{G})$ of periodic automorphisms with bounded periods.

To see that $\|\cdot\|$ is large-scale geodesic (with constant $K = 2$), let us take a non-trivial $T \in \mathfrak{D}(\mathbb{G})$ and pick a periodic $U \in \mathfrak{D}(\mathbb{G})$ with bounded periods such that $\left\|TU^{-1}\right\| < \min\{2, \|T\|/2\}$. Note that

$$\|U\| = \|U^{-1}\| = \|T^{-1}TU^{-1}\| \leq \|T^{-1}\| + \|TU^{-1}\| \leq 3\|T\|/2 \tag{3.3}$$

Fix $n \in \mathbb{N}$ large enough to ensure $\frac{3\|T\|}{2n} \leq 2$. By Lemma 3.24, we may decompose U into a product of n elements $U_1, \ldots, U_n$ each of norm at most $\frac{3\|T\|}{2n} \leq 2$. Therefore

$$T = (TU^{-1}) \cdot U_1 \cdots U_n,$$

where TU^{-1} and each of U_i, $1 \leq i \leq n$, has norm at most 2 and, in view of Eq. (3.3),

$$\|TU^{-1}\| + \sum_{i=1}^{n} \|U_i\| \leq \frac{\|T\|}{2} + \|U\| \leq 2\|T\|,$$

thus concluding the proof that $\|\cdot\|$ is large-scale geodesic.

We now show that $\|\cdot\|$ is coarsely proper. Fix $\epsilon > 0$ and $R > 0$. Let $n \in \mathbb{N}$ be so large that $n\epsilon \geq R + \epsilon$. Then every element $T \in \mathfrak{D}(\mathbb{G})$ of norm at most R is a product of $n + 1$ elements of norm at most ϵ, namely one element TU^{-1} of norm at most ϵ, where U is periodic with bounded periods as provided by density, and $U = U_1 \cdots U_n$, where each U_i has norm at most $\frac{R+\epsilon}{n} \leq \epsilon$ as per Lemma 3.24. Thus $\|\cdot\|$ is both coarsely proper and large-scale geodesic, and hence is maximal by Proposition A.10. ∎

Remark 3.26. We do not have an example of an induction friendly Polish finitely full group $\mathbb{G}$ for which the periodic elements are not dense in $\mathfrak{D}(\mathbb{G})$. A potential candidate might be the L^1 full group of a free action of the free group on 2 generators, endowed with the norm given by the word length with respect to the canonical generating set.

Chapter 4

Full groups of locally compact group actions

In this chapter, we narrow down the generality of the narrative and focus on actions of *locally compact* Polish groups, or equivalently, of locally compact second-countable groups. Such restrictions enlarge our toolbox in a number of ways. For instance, all locally compact Polish group actions admit cross-sections to which the so-called Voronoi tessellations can be associated. We use this to show in Section 4.1 a natural density result for subsets of L^1 full groups defined from dense subsets of the acting group (Theorem 4.2 and Corollary 4.3). For the reader's convenience, Appendix E.2 contains a concise reminder of the needed facts about tessellations.

Another key property of free[1] actions of locally compact groups is the existence of a Haar measure on each individual orbit. As we discuss in Section 4.2, elements of the full group act by non-singular transformations and, in particular, admit the Hopf decomposition (see Appendix C). Section 4.3 explains how these orbitwise decompositions can be understood globally, yielding a natural generalization of the periodic/aperiodic partition for elements of the full group of a measure-preserving action of a *discrete* group. The periodic part in the latter case corresponds to the conservative piece of the Hopf decomposition, which generally exhibits a much more complicated dynamical behavior. We return to this in Chapters 7 and 8.

In the final Section 4.4, we connect L^1 full groups to the notion of L^1 orbit equivalence for actions of locally compact *compactly generated* Polish groups.

4.1 Dense subgroups in L^1 full groups

Our goal in this section is to prove that any element of the full group $[G \curvearrowright X]$ can be approximated arbitrarily well by an automorphism that piecewise acts by elements of a given dense subset of G.

Definition 4.1. A measure-preserving transformation $T : A \to B$ between two measurable sets $A, B \subseteq X$ is said to be H**-decomposable**, where $H \subseteq \mathrm{Aut}(X, \mu)$, if there exist a measurable partition $A = \bigsqcup_{k\in\mathbb{N}} A_k$ and elements $h_k \in H$ such that $T \restriction_{A_k} = h_k \restriction_{A_k}$ for all $k \in \mathbb{N}$.

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[1] Motivated by our focus on $\mathbb{R}$-flows, this monograph primarily concentrates on *free* actions. We note, however, that each orbit of a Borel action of a locally compact Polish group is a homogeneous space, since point stabilizers are necessarily closed. In particular, orbits can be endowed with the Haar measure, even without the freeness assumption.

The property of being H-decomposable is similar to being an element of the full group generated by H, except that we do not require the transformation to be defined on all of X.

Theorem 4.2. *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish group. Let $\|\cdot\|$ be a compatible norm on G with the associated metric on the orbits $D : \mathcal{R}_G \to \mathbb{R}^{\geq 0}$, and let $H \subseteq G$ be a dense set. For any $T \in [G \curvearrowright X]$ and any $\epsilon > 0$, there exists an H-decomposable transformation $S \in [G \curvearrowright X]$ such that* $\operatorname{ess\,sup}_{x \in X} D(Tx, Sx) < \epsilon$.

Theorem 4.2 establishes the density of H-decomposable transformations in the very strong uniform topology given by ess sup. In particular, this result also applies to the L^1 topology.

Corollary 4.3. *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish group, let $\|\cdot\|$ be a compatible norm on G, and let $H \subseteq G$ be a dense subgroup. The* L^1 *full group $[H \curvearrowright X]_1$ is dense in $[G \curvearrowright X]_1$.*

Remark 4.4. Theorem 4.2 is an improvement upon the conclusion of [12, Thm. 2.1], which shows that $[H \curvearrowright X]$ is dense in $[G \curvearrowright X]$ whenever H is a dense subgroup of G. While the proof, which we present below, establishes density in a much stronger topology through more elementary means, we note that, as already mentioned in [12, Thm. 2.3], their methods apply to all *suitable* (in the sense of [5]; see also Definition 4.7) actions of Polish groups, whereas our approach here crucially uses local compactness of the acting group to guarantee existence of various cross-sections.

Let C be a cross-section for a measure-preserving action $G \curvearrowright X$, and let $\mathcal{W}$ be a tessellation over C (in the sense of Appendix E.2). Let $\nu_{\mathcal{W}}$ be the push-forward measure $(\pi_{\mathcal{W}})_*\mu$ on the cross-section, and let $(\mu_c)_{c\in C}$ be the disintegration of μ over $(\pi_{\mathcal{W}}, \nu_{\mathcal{W}})$ (see Appendix D and Theorem D.1, specifically). Without loss of generality, we assume, whenever convenient, that the set H in the statement of Theorem 4.2 is countable.

Definition 4.5. Two Borel sets $A, B \subseteq X$ are said to be

- **$\mathcal{W}$-proportionate** if the equivalence $\mu_c(A) = 0 \iff \mu_c(B) = 0$ holds for $\nu_{\mathcal{W}}$-almost all $c \in C$;
- **$\mathcal{W}$-equimeasurable** if $\mu_c(A) = \mu_c(B)$ for $\nu_{\mathcal{W}}$-almost all $c \in C$.

For the context of Lemmas 4.6 through 4.11, we let N denote an open *symmetric* neighborhood of the identity of G, and $\mathcal{W}$ stands for an N-lacunary tessellation. The following lemma relies on the key fact that for any two $\mathcal{W}$-proportionate Borel sets $A, B \subseteq N \cdot C$, the equivalence $A \cap (N \cdot c) \neq \varnothing \iff B \cap (N \cdot c) \neq \varnothing$ holds for all $c \in C$ after changing A and B on a null set.

Lemma 4.6. *If $A, B \subseteq N \cdot C$ are $\mathcal{W}$-proportionate Borel sets then*

$$\mu(B \setminus N^2 \cdot A) = 0.$$

Proof. By the defining property of the disintegration,

$$\mu(B \setminus N^2 \cdot A) = \int_C \mu_c(B \setminus N^2 \cdot A)\, d\nu_{\mathcal{W}}(c),$$

and so we need to check that $\mu_c(B \setminus N^2 \cdot A) = 0$ for $\nu_{\mathcal{W}}$-almost all c. Since A and B are $\mathcal{W}$-proportionate, it suffices to show that $\mu_c(B \setminus N^2 \cdot A) = 0$ whenever $\mu_c(A) \neq 0$. For any $c \in C$ satisfying the latter, one necessarily has $c \in N \cdot A$ (because $A \subseteq N \cdot C$ and $\mathcal{W}$ is N-lacunary, by assumption), and thus $N \cdot c \subseteq N^2 \cdot A$. In particular, $(B \setminus N^2 \cdot A) \cap N \cdot c = \varnothing$. It remains to use the inclusion $B \subseteq N \cdot C$, which, together with the N-lacunarity of $\mathcal{W}$, guarantees that

$$\mu_c(B \setminus N^2 \cdot A) = \mu_c((B \setminus N^2 \cdot A) \cap N \cdot c) = 0. \qquad \blacksquare$$

For the proof of the next lemma, we need the notion of a suitable action, introduced by H. Becker [5, Def. 1.2.7].

Definition 4.7. A measure-preserving Borel action $G \curvearrowright X$ of a Polish group G is **suitable** if for all Borel sets $A, B \subseteq X$ one of the following two options holds:

(1) for any open neighborhood of the identity $M \subseteq G$ there exists $g \in M$ such that $\mu(gA \cap B) > 0$;

(2) there exist Borel sets $A' \subseteq A$, $B' \subseteq B$ such that $\mu(A \setminus A') = 0 = \mu(B \setminus B')$ and an open neighborhood of the identity $M \subseteq G$ such that $M \cdot A' \cap B' = \varnothing$.

All measure-preserving actions of locally compact Polish groups are known to be suitable (see [5, Thm. 1.2.9]).

Lemma 4.8. *For all non-negligible $\mathcal{W}$-proportionate Borel sets $A, B \subseteq N \cdot C$, there exists an open set $U \subseteq N^3$ such that $\mu(gA \cap B) > 0$ for all $g \in U$.*

Proof. Let $H_1 = \{h_n : n \in \mathbb{N}\}$ be a countable dense subset of $N^2 = NN^{-1}$, and put $A_1 = H_1 \cdot A$. We apply the dichotomy in the definition of a suitable action to the sets A_1, B and show that item (2) cannot hold.

Indeed, suppose there exist $A_1' \subseteq A_1$, $B' \subseteq B$ satisfying

$$\mu(A_1 \setminus A_1') = 0 = \mu(B \setminus B'),$$

and an open neighborhood of the identity $M \subseteq G$ such that $(M \cdot A_1') \cap B' = \varnothing$. Set $A' = \bigcap_n (h_n^{-1} A_1' \cap A)$, and note that $\mu(A \setminus A') = 0$ and $(MH_1 \cdot A') \cap B' = \varnothing$, simply because $C_1 \cdot A' \subseteq A_1'$. Since C_1 is dense in N^2, we have $N^2 \subseteq MH_1$ and thus $(N^2 \cdot A') \cap B' = \varnothing$.

Lemma 4.6, applied to A' and B', guarantees that $\mu(B' \setminus N^2 \cdot A') = 0$, which is possible only when $\mu(B') = 0$, contradicting the assumption that B is non-negligible.

We are left with the alternative of the item (1), and so there has to exist some $g \in N$ such that $\mu(gA_1 \cap B) > 0$. Since $A_1 = H_1 \cdot A$, there exists $h \in H_1$ such that $\mu(ghA \cap B) > 0$. It remains to observe that $gh \in N^3$ and that $\mu(g'A \cap B) > 0$ is an open condition on g'. This follows from the continuity in the weak topology of the group homomorphism $G \to \mathrm{Aut}(X, \mu)$ associated with the measure-preserving action of G on (X, μ) (see, for instance, [12, Lem. 1.2]). ■

Lemma 4.9. *For any non-empty open $V \subseteq N$ and for any non-negligible Borel set $A \subseteq X$, there exists $h \in H$ such that*

$$\mu(\{x \in A : hx \in V \cdot C \textit{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(hx)\}) > 0.$$

Proof. Let $\zeta : X \to \mathcal{W}$ be the Borel bijection $\zeta(x) = (\pi_{\mathcal{W}}(x), x)$ and consider the push-forward measure $\zeta_*\mu$, which for $Z \subseteq \mathcal{W}$ can be expressed as

$$\zeta_*\mu(Z) = \int_C \mu_c(Z_c)\, d\nu_{\mathcal{W}}(c).$$

Let $(h_n)_{n \in \mathbb{N}}$ be an enumeration of H and set

$$W_n = \{(c, x) \in \mathcal{W} : \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_n x) \text{ and } h_n x \in V \cdot C\}.$$

We claim that $\bigcup_n W_n = \mathcal{W}$. Indeed, for each $(c, x) \in \mathcal{W}$ the set of $g \in G$ such that $gx \in V \cdot c$ is non-empty and open, hence there is $h_n \in H$ such that $h_n x \in V \cdot c$.

Finally, A is non-negligible by assumption, i.e., $0 < \mu(A) = \zeta_*\mu(\zeta(A))$, so there exists W_n such that $\zeta_*\mu(\zeta(A) \cap W_n) > 0$, which translates into the required

$$\mu(\{x \in A : h_n x \in V \cdot C \text{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_n x)\}) > 0. \qquad ■$$

Lemma 4.10. *For all non-negligible $\mathcal{W}$-proportionate Borel sets $A, B \subseteq X$, there exists $h \in H$ such that*

$$\mu(\{x \in A : hx \in B \textit{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(hx)\}) > 0.$$

Proof. The plan is to reduce the setup of this lemma to that of Lemma 4.8. Let $V \subseteq N$ be a symmetric neighborhood of the identity that is furthermore small enough to guarantee that $\mathcal{W}$ is V^4-lacunary. Apply Lemma 4.9 to find $h_1 \in H$ such that for

$$A' = \{x \in A : h_1 x \in V \cdot C \text{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_1 x)\}$$

one has $\mu(A') > 0$. Set $A_1 = h_1 A'$, $B_1 = \pi_{\mathcal{W}}^{-1}(\{c \in C : \mu_c(A_1) > 0\}) \cap B$ and note that A_1 and B_1 are non-negligible $\mathcal{W}$-proportionate sets. Moreover, $A_1 \subseteq V \cdot C$ by construction.

Repeat the same steps for B_1 and find $h_2 \in H$ such that for

$$B_1' = \{x \in B_1 : h_2 x \in V \cdot C \text{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_2 x)\}$$

we have $\mu(B_1') > 0$. Set $B_2 = h_2 B_1'$ and $A_2 = A_1 \cap \pi_{\mathcal{W}}^{-1}(\{c \in C : \mu_c(B_2) > 0\})$. Once again, sets A_2 and B_2 are non-negligible, $\mathcal{W}$-proportionate and are both contained in $V \cdot C$.

We now apply Lemma 4.8 to sets A_2, B_2 and $\mathcal{W}$, viewed as a V-lacunary tessellation, yielding an open $U \subseteq V^3$ such that $\mu(gA_2 \cap B_2) > 0$ for all $g \in U$. Note that since $U \subseteq V^3$ and $\mathcal{W}$ is, in fact, V^4-lacunary, the equality $\pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(gx)$ holds for all $x \in V \cdot C$ and $g \in U$. We conclude that $\mu(h_2^{-1} g h_1 A \cap B) > 0$ for all $g \in U$ and hence any $h \in h_2^{-1} U h_1 \cap H$ satisfies the conclusion of the lemma. ■

A measure-preserving partial transformation $T : A \to B$ is **$\mathcal{W}$-coherent** if μ-almost surely one has $\pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(Tx)$.

Lemma 4.11. *For all $\mathcal{W}$-equimeasurable Borel sets $A, B \subseteq X$, there exists a $\mathcal{W}$-coherent H-decomposable measure-preserving bijection $T : A \to B$.*

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be an enumeration of H. Consider the set

$$A_0 = \big\{x \in A : h_0 x \in B \text{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_0 x)\big\},$$

and let $B_0 = h_0 A_0$. Note that the sets $A \setminus A_0$ and $B \setminus B_0$ are $\mathcal{W}$-equimeasurable, so we may continue in the same fashion and construct sets A_k such that

$$A_k = \Big\{x \in A \setminus \bigsqcup_{i<k} A_i : h_k x \in B \setminus \bigsqcup_{i<k} B_i \text{ and } \pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(h_k x)\Big\}.$$

We define $T : \bigsqcup_{k\in\mathbb{N}} A_k \to \bigsqcup_{k\in\mathbb{N}} B_k$ by the condition $Tx = h_k x$ for $x \in A_k$.

Sets $A \setminus \bigsqcup_{k\in\mathbb{N}} A_k$ and $B \setminus \bigsqcup_{k\in\mathbb{N}} B_k$ are $\mathcal{W}$-equimeasurable. If either one of them (and thus necessarily both of them) were non-negligible, Lemma 4.10 would yields an element $h \in H$ that moves a portion of $A \setminus \bigsqcup_{k\in\mathbb{N}} A_k$ into $B \setminus \bigsqcup_{k\in\mathbb{N}} B_k$, contradicting the construction. We conclude that

$$\mu(A \setminus \bigsqcup_{k\in\mathbb{N}} A_k) = 0 = \mu(B \setminus \bigsqcup_{k\in\mathbb{N}} B_k)$$

and T is therefore as required. ■

Lemma 4.12. *Suppose that $\mathcal{W}$ is a cocompact tessellation over the cross-section C. Let $A, B \subseteq X$ be $\mathcal{W}$-equimeasurable Borel sets. For any $\epsilon > 0$, and any $\mathcal{W}$-coherent partial transformation $T : A \to B$, there exists a $\mathcal{W}$-coherent H-decomposable $\tilde{T} : A \to B$ such that* $\operatorname{ess\,sup}_{x\in A} D(Tx, \tilde{T}x) < \epsilon$.

Proof. Let $\mathcal{V}$ be a K'-cocompact tessellation over some cross-section C' such that the diameter of each region in $\mathcal{V}$ is less than ϵ. Suppose $\mathcal{W}$ is K-cocompact. By Lemma E.2, we can find a finite partition of $C' = \bigsqcup_{i \leq n} C'_i$ such that each C'_i is $K'K^2K'$-lacunary, which guarantees that, for each i, every $\mathcal{W}_c$ intersects at most one class $\mathcal{V}_{c'}$, $c' \in C'_i$. For each $i, j < n$ set $A_{(i,j)} = \{x \in A : \pi_{\mathcal{V}}(x) \in C'_i, \pi_{\mathcal{V}}(Tx) \in C'_j\}$ and $B_{(i,j)} = TA_{(i,j)}$. We re-enumerate sets $A_{(i,j)}$ and $B_{(i,j)}$ as a sequence A_k, B_k, $k \leq n^2$ and note that for all $x, y \in A_k$ one has

$$\pi_{\mathcal{W}}(x) = \pi_{\mathcal{W}}(y) \implies \bigl(\pi_{\mathcal{V}}(x) = \pi_{\mathcal{V}}(y) \text{ and } \pi_{\mathcal{V}}(Tx) = \pi_{\mathcal{V}}(Ty)\bigr).$$

Moreover, sets A_k and $T(A_k)$ are $\mathcal{W}$-equimeasurable, so Lemma 4.11 yields $\mathcal{W}$-coherent H-decomposable partial transformations $T_k : A_k \to T(A_k)$. The transformation $\tilde{T} : A \to B$ can now be defined by the condition $\tilde{T}x = T_k x$ whenever $x \in A_k$. It is easy to check that $\tilde{T}$ is as claimed. ∎

Proof of Theorem 4.2. By Proposition E.4, we have a sequence $(\mathcal{V}_k)_k$ of cocompact tessellations such that $\mathcal{R}_G = \bigcup_k \mathcal{R}_{\mathcal{V}_k}$. Let $A_0 = \{x \in X : \pi_{\mathcal{V}_0}(x) = \pi_{\mathcal{V}_0}(Tx)\}$. Use Lemma 4.12 to find an H-decomposable partial transformation $T_0 : A_0 \to T(A_0)$ that satisfies the inequality $\operatorname{ess\,sup}_{x \in A_0} D(T_0 x, Tx) < \epsilon$. Set

$$A_k = \Bigl\{x \in X : \pi_{\mathcal{V}_k}(x) = \pi_{\mathcal{V}_k}(Tx) \text{ and } x \notin \bigsqcup_{l<k} A_l\Bigr\}$$

and note that A_k, $k \in \mathbb{N}$, form a partition of X because $\mathcal{R}_G = \bigcup_k \mathcal{R}_{\mathcal{V}_k}$. Construct partial transformations $T_k : A_k \to T(A_k)$ via repeated applications of Lemma 4.12 to the tessellations $\mathcal{V}_k$. The element $S \in [G \curvearrowright X]$ defined for $x \in A_k$ by $Sx = T_k x$ satisfies the conclusion of the theorem. ∎

4.2 Orbital transformations

Let $G \curvearrowright X$ be a *free* measure-preserving action of a locally compact Polish group on a standard probability space. Recall that the identification of G with its orbits induces the cocycle map $\rho : \mathcal{R}_G \to G$, defined by $\rho(x, gx) = g$. Moreover, every $T \in [\mathcal{R}_G]$ has an associated cocycle $\rho_T : X \to G$ determined by the condition $T(x) = \rho_T(x)x$ for all $x \in X$.

Fix a *right*-invariant Haar measure λ on G. Since any orbit $[x]_{\mathcal{R}_G}$ can be identified with the group G via the map $G \ni g \mapsto gx \in [x]_{\mathcal{R}_G}$, the measure λ can be pushed forward through this identification to define a collection of measures $(\lambda_x)_{x \in X}$ on X. These measures are given by $\lambda_x(A) = \lambda(\{g \in G : gx \in A\})$. The right invariance of λ ensures that λ_x depends only on the orbit $[x]_{\mathcal{R}_G}$ and is independent of the choice of the base point; that is, $\lambda_x = \lambda_y$ whenever $x \mathcal{R}_G y$.

This section focuses on two main facts: the so-called mass-transport principle, given in Eq. (4.1) below, and the non-singularity of the transformations induced by elements of $[G \curvearrowright X]$ onto orbits of the action, formulated in Proposition 4.13. Both of these topics have been discussed in the literature in various related contexts, including, for instance, [12, Appen. A] and the treatise [2]. However, we are not aware of any specific reference from which Eq. (4.1) and Proposition 4.13 can be readily deduced. The following derivations are therefore included for the reader's convenience, with the disclaimer that these results are likely to be known to experts.

The freeness of the action allows us to identify the equivalence relation $\mathcal{R}_G$ with $X \times G$ via $\Phi : X \times G \to \mathcal{R}_G$, $\Phi(x, g) = (x, gx)$. The push-forward $\Phi_*(\mu \times \lambda)$ of the product measure is denoted by M and can equivalently be defined by

$$M(A) = \int_X \lambda_x(A_x)\, d\mu(x),$$

where $A \subseteq \mathcal{R}_G$ and $A_x = \{y \in X : (x, y) \in A\}$.

In general, the flip transformation $\sigma : \mathcal{R}_G \to \mathcal{R}_G$, $\sigma(x, y) = (y, x)$, does not preserve the measure M. Set $\Psi : X \times G \to X \times G$ to be the involution $\Psi = \Phi^{-1} \circ \sigma \circ \Phi$, which simplifies to $\Psi(x, g) = (gx, g^{-1})$. Following the computation as in [12, Prop. A.11], one can easily check that $\Psi_*(\mu \times \lambda) = \mu \times \widehat{\lambda}$, where $\widehat{\lambda}$ is the associated *left*-invariant measure, $\widehat{\lambda}(A) = \lambda(A^{-1})$. If we define the measure $\widehat{M}$ on $\mathcal{R}_G$ to be

$$\widehat{M}(A) = \Phi_*(\mu \times \widehat{\lambda}) = \int_X \widehat{\lambda}_x(A_x)\, d\mu(x),$$

then $\sigma_* M = \widehat{M}$, and also $\sigma_* \widehat{M} = M$, since $\sigma^{-1} = \sigma$. In particular, σ is M-invariant if and only if $\lambda = \widehat{\lambda}$, i.e., G is unimodular.

A function $f : \mathcal{R}_G \to \mathbb{R}$ is M-integrable if and only if $X \times G \ni (x, g) \mapsto f(x, gx)$ is $(\mu \times \lambda)$-integrable. Using Fubini's theorem and noting that $\Phi \circ \Psi^{-1} = \Phi \circ \Psi = \sigma \circ \Phi$, we get the following chain of identities for such a function f:

$$\begin{aligned}
\int_X \int_G f(x, gx)\, d\lambda(g) d\mu(x) &= \int_{X \times G} f(x, gx)\, d(\mu \times \lambda)(x, g) \\
&= \int_{X \times G} f \circ \Phi\, d(\Psi_*^{-1}(\mu \times \widehat{\lambda})) \\
&= \int_{X \times G} f \circ \Phi \circ \Psi^{-1}\, d(\mu \times \widehat{\lambda}) \\
&= \int_{X \times G} f \circ \sigma \circ \Phi\, d(\mu \times \widehat{\lambda}) \\
\int_X \int_G f(x, gx)\, d\lambda(g) d\mu(x) &= \int_X \int_G f(gx, x)\, d\widehat{\lambda}(g) d\mu(g).
\end{aligned}$$

Let $\Delta : G \to \mathbb{R}^{>0}$ be the *left Haar modulus* given for $g \in G$ by $\lambda(gA) = \Delta(g)\lambda(A)$. Recall that $\Delta : G \to \mathbb{R}^{>0}$ is a continuous homomorphism (see [47, Prop. 7]). The measures λ and $\widehat{\lambda}$ belong to the same measure class, with the Radon–Nikodym derivative $\frac{d\widehat{\lambda}}{d\lambda}(g) = \Delta(g^{-1})$ for all $g \in G$ (see [47, p. 79]). The identities above translate into the following:

$$\int_X \int_G f(x, g \cdot x)\, d\lambda(g) d\mu(x) = \int_X \int_G \Delta(g^{-1}) f(g \cdot x, x)\, d\lambda(g) d\mu(x). \tag{4.1}$$

When the group G is unimodular, this expression attains a very symmetric form and is known as the **mass-transport principle**:

$$\int_X \int_G f(x, g \cdot x)\, d\lambda(g) d\mu(x) = \int_X \int_G f(g \cdot x, x)\, d\lambda(g) d\mu(x). \tag{4.2}$$

Any automorphism $T \in [G \curvearrowright X]$ induces, for each $x \in X$, a transformation of the σ-finite measure space (X, λ_x). In general, T does not preserve λ_x; however, it is always non-singular, and the Radon–Nikodym derivative $\frac{dT_*\lambda_x}{d\lambda_x}$ can be described explicitly. Note that the full group $[G \curvearrowright X]$ admits two natural actions on the equivalence relation $\mathcal{R}_G$: the *left* action l is given by $l_T(x, y) = (Tx, y)$, and the *right* action r is defined as $r_T(x, y) = (x, Ty)$. A straightforward verification (see [12, Lem. A.9]) shows that l is always M-invariant. Since $r_T \circ \sigma = \sigma \circ l_T$, for all $T \in [G \curvearrowright X]$, we have

$$(r_T)_* \widehat{M} = (r_T \circ \sigma)_* M = (\sigma \circ l_T)_* M = \sigma_* M = \widehat{M}.$$

Let $\Theta = \Phi^{-1} \circ r_T \circ \Phi$, i.e., $\Theta(x, g) = (x, \rho_{Tg}(x))$. The equality $(r_T)_* \widehat{M} = \widehat{M}$ is equivalent to $\Theta_*(\mu \times \widehat{\lambda}) = \mu \times \widehat{\lambda}$. The latter implies that for each Borel set $B \subseteq G$ and all measurable sets $A \subseteq X$, we have

$$\begin{aligned}
\int_A \widehat{\lambda}(B)\, d\mu &= (\mu \times \widehat{\lambda})(A \times B) = \Theta_*(\mu \times \widehat{\lambda})(A \times B) \\
&= (\mu \times \widehat{\lambda})\big(\{(x, g) \in X \times G : (x, \rho_{Tg}(x)) \in A \times B\}\big) \\
\text{Fubini's theorem } &= \int_A \widehat{\lambda}(\{g \in G : \rho_{Tg}(x) \in B\})\, d\mu(x) \\
&= \int_A \widehat{\lambda}(\{g \in G : gx \in T^{-1}Bx\})\, d\mu(x),
\end{aligned}$$

which is possible only if $\widehat{\lambda}(\{g \in G : gx \in T^{-1}Bx\}) = \widehat{\lambda}(B)$ for μ-almost all x. Passing to the measures on the orbits, this translates for each B into $\widehat{\lambda}_x(T^{-1}Bx) = \widehat{\lambda}_x(Bx)$. If $(B_n)_{n \in \mathbb{N}}$ is a countable algebra of Borel sets in G that generates the whole Borel σ-algebra, then for each $x \in X$, $(B_n x)_{n \in \mathbb{N}}$ is an algebra of Borel subsets of the orbit $[x]_{\mathcal{R}_G}$, which generates the Borel σ-algebra on it. We have established that for μ-almost all $x \in X$, the two measures, $\widehat{\lambda}_x$ and $T_* \widehat{\lambda}_x$, coincide on each $B_n x$, $n \in \mathbb{N}$, thus μ-almost surely $\widehat{\lambda}_x = T_* \widehat{\lambda}_x$.

The equality $\frac{d\widehat{\lambda}}{d\lambda}(g) = \Delta(g^{-1})$ translates into $\frac{d\widehat{\lambda}_x}{d\lambda_x}(y) = \Delta(\rho(x,y)^{-1}) = \Delta(\rho(y,x))$, and the Radon–Nikodym derivative $\frac{dT_*\lambda_x}{d\lambda_x}$ can now be computed as follows:

$$\begin{aligned} \frac{dT_*\lambda_x}{d\lambda_x}(y) &= \frac{dT_*\lambda_x}{dT_*\widehat{\lambda}_x}(y) \cdot \frac{dT_*\widehat{\lambda}_x}{d\widehat{\lambda}_x}(y) \cdot \frac{d\widehat{\lambda}_x}{d\lambda_x}(y) \\ T \text{ preserves } \widehat{\lambda}_x &= \frac{dT_*\lambda_x}{dT_*\widehat{\lambda}_x}(y) \cdot \frac{d\widehat{\lambda}_x}{d\lambda_x}(y) = \frac{d\lambda_x}{d\widehat{\lambda}_x}(T^{-1}y) \cdot \frac{d\widehat{\lambda}_x}{d\lambda_x}(y) \\ &= \left(\frac{d\widehat{\lambda}_x}{d\lambda_x}(T^{-1}y)\right)^{-1} \cdot \frac{d\widehat{\lambda}_x}{d\lambda_x}(y) \\ &= \Delta(\rho(x,T^{-1}y)^{-1})^{-1}\Delta(\rho(x,y)^{-1}) \\ &= \Delta\big(\rho(x,T^{-1}y) \cdot \rho(y,x)\big) = \Delta(\rho_{T^{-1}}(y)). \end{aligned}$$

We summarize the content of this section into a proposition.

Proposition 4.13. *Let G be a locally compact Polish group acting freely $G \curvearrowright X$ on a standard probability space (X,μ). Let λ be a right Haar measure on G, $\Delta : G \to \mathbb{R}^{>0}$ be the corresponding Haar modulus, and let $(\lambda_x)_{x\in X}$ be the family of measures obtained by pushing λ onto orbits via the action map. Each $T \in [G \curvearrowright X]$ induces a non-singular transformation of (X,λ_x) for almost every $x \in X$, and moreover, one has $\lambda_x(T^{-1}A) = \int_A \Delta(\rho_{T^{-1}}(y))\, d\lambda_x(y)$ for all Borel sets $A \subseteq X$. If G is unimodular, then $T_*\lambda_x = \lambda_x$ for μ-almost all $x \in X$.*

For future reference, we isolate a simple lemma, which is an immediate consequence of Fubini's theorem.

Lemma 4.14. *Let G be a locally compact Polish group acting freely on a standard probability space (X,μ). Let λ, $\widehat{\lambda}$, $(\lambda_x)_{x\in X}$, and $(\widehat{\lambda}_x)_{x\in X}$ be as above. For any Borel set $A \subseteq X$, the following are equivalent:*

(1) *$\mu(A) = 0$;*

(2) *$\lambda_x(A) = 0$ for μ-almost all $x \in X$;*

(3) *$\widehat{\lambda}_x(A) = 0$ for μ-almost all $x \in X$.*

Proof. (1) $\iff$ (2) Using Fubini's Theorem on $(X \times G, \mu \times \lambda)$ to rearrange the order of quantifiers, one has:

$$\begin{aligned} \mu(A) = 0 \iff & \forall g \in G\ \forall^\mu x \in X\ gx \notin A \\ & \iff \forall^\mu x \in X\ \forall^\lambda g \in G\ gx \notin A \iff \forall^\mu x \in X\ \lambda_x(A) = 0. \end{aligned}$$

(2) $\iff$ (3) is evident, since λ and $\widehat{\lambda}$ are equivalent measures, hence so are λ_x and $\widehat{\lambda}_x$ for all $x \in X$. ∎

4.3 The Hopf decomposition of elements of the full group

Fix an element $T \in [G \curvearrowright X]$ of the full group of a free measure-preserving action of a locally compact Polish group G. As explained in Section 4.2, T acts naturally in a non-singular manner on each G-orbit. This action thus has a Hopf decomposition (see Appendix C). We will now explain how to interpret this decomposition globally, thereby generalizing the fact that, when G is discrete, any element of the full group decomposes the space into periodic and aperiodic parts.

Let C be a cocompact cross-section, and let $\mathcal{V}_C$ be the Voronoi tessellation associated with some proper norm on G (see Appendix E.2). Set $\pi_C : X \to C$ to be the projection map given by the condition $(\pi_C(x), x) \in \mathcal{V}_C$ for all $x \in X$. The dissipative and conservative sets of the transformation T are defined as follows:

$$\begin{aligned}D_T &= \bigl\{x \in X : \exists n \in \mathbb{N}\ \forall k \in \mathbb{Z} \text{ such that } |k| \geq n \text{ one has } \pi_C(x) \neq \pi_C(T^k x)\bigr\},\\ C_T &= \bigl\{x \in X : \forall n \in \mathbb{N}\ \exists k_1, k_2 \in \mathbb{Z} \text{ such that}\\ &\qquad\qquad k_1 \leq -n, n \leq k_2 \text{ and } \pi_C(T^{k_1}x) = \pi_C(x) = \pi_C(T^{k_2}x)\bigr\}.\end{aligned}$$

In plain words, the dissipative set D_T consists of those points x whose orbit has a finite intersection with the Voronoi region of x. The conservative set C_T, on the other hand, collects all the points whose orbit is bi-recurrent in the region. We argue in Proposition 4.16 that the sets D_T and C_T induce the Hopf decomposition for $T\restriction_{[x]_{\mathcal{R}_T}}$ for almost every $x \in X$; in particular, $D_T \sqcup C_T$ is a partition of X, which is independent of the choice of the cross-section C.

Lemma 4.15. *The sets D_T and C_T partition the phase space: $X = D_T \sqcup C_T$.*

Proof. Define sets N_+ and N_- according to

$$\begin{aligned}N_+ &= \{x \in X \setminus (D_T \sqcup C_T) : \forall k \geq 1\ \pi_C(T^k x) \neq \pi_C(x)\},\\ N_- &= \{x \in X \setminus (D_T \sqcup C_T) : \forall k \geq 1\ \pi_C(T^{-k} x) \neq \pi_C(x)\},\end{aligned}$$

and note that $X \setminus (D_T \sqcup C_T) \subseteq \bigcup_{k \in \mathbb{Z}} T^k(N_+ \cup N_-)$. To show that $X = D_T \sqcup C_T$ it is enough to verify that $\mu(N_+) = 0 = \mu(N_-)$.

This is done by noting that these sets admit pairwise disjoint copies using piecewise translations by powers of T. In view of the fact that T is measure-preserving, this implies that N_+ and N_- are null. To be more precise, set $N_-^0 = N_-$ and define inductively $N_-^n = \{T^{k(x)}x : x \in N_-^{n-1}\}$, where $k(x) \geq 1$ is the smallest natural number such that $\pi_C(T^{k(x)}x) = \pi_C(x)$. Note that $k(x)$ is well-defined, for otherwise x would belong to D_T. Sets N_-^n, $n \in \mathbb{N}$, are pairwise disjoint, and have the same measure since T is measure-preserving. We conclude that $\mu(N_-) = 0$. The argument for $\mu(N_+) = 0$ is similar. ∎

Proposition 4.16 (Hopf decomposition). *Let $G \curvearrowright X$ be a free measure-preserving action of a locally compact Polish group on a standard probability space (X, μ). Let λ be a right Haar measure on G and $(\lambda_x)_{x \in X}$ be the push-forward of λ onto the orbits as described in Section 4.2. For any element $T \in [G \curvearrowright X]$, the measurable T-invariant partition $X = D_T \sqcup C_T$ defined above satisfies that for μ-almost all $x \in X$ the partition $[x]_{\mathcal{R}_G} = ([x]_{\mathcal{R}_G} \cap D_T) \sqcup ([x]_{\mathcal{R}_G} \cap C_T)$ is the Hopf decomposition for $T \restriction_{[x]_{\mathcal{R}_G}}$ on $([x]_{\mathcal{R}_G}, \lambda_x)$. Moreover, there is only one partition $X = D_T \sqcup C_T$ satisfying this property up to null sets.*

Proof. According to Proposition 4.13, we may assume that for all $x \in X$ the map $T \restriction_{[x]_{\mathcal{R}_G}} : [x]_{\mathcal{R}_G} \to [x]_{\mathcal{R}_G}$ is a non-singular transformation with respect to λ_x and satisfies $\lambda_x(TA) = \int_A \Delta(\rho_T(y))\, d\lambda_x(y)$ for all Borel $A \subseteq X$.

Let $[x]_{\mathcal{R}_G} = D_x \sqcup C_x$, $x \in X$, denote the Hopf decomposition for $T \restriction_{[x]_{\mathcal{R}_G}}$. For any $c \in C$, the set

$$\widetilde{W}_c = \left\{x \in (\mathcal{V}_C)_c : T^k x \notin (\mathcal{V}_C)_c \text{ for all } k \geq 1\right\}$$

is a wandering set and therefore $\widetilde{W}_c \subseteq D_x$ up to a null set. If $x \in D_T$ satisfies $x \in (\mathcal{V}_C)_c$, $c \in C$, then $[x]_{\mathcal{R}_G} \cap (\mathcal{V}_C)_c$ is finite, and therefore $[x]_{\mathcal{R}_G} \cap (\mathcal{V}_C)_c \subseteq \bigcup_{k \in \mathbb{Z}} T^k \widetilde{W}_c$, whence also

$$[x]_{\mathcal{R}_G} \cap D_T \subseteq \bigcup_{c \in C \cap [x]_{\mathcal{R}_G}} \bigcup_{k \in \mathbb{Z}} T^k \widetilde{W}_c \subseteq D_x.$$

Claim. We have $\lambda_x([x]_{\mathcal{R}_G} \cap C_T \cap D_x) = 0$ for each $x \in X$.

Proof of the claim. Otherwise we can find $c \in C \cap [x]_{\mathcal{R}_G}$ and a wandering set $W \subseteq [x]_{\mathcal{R}_G} \cap (\mathcal{V}_C)_c \cap C_T$ of positive measure, $\lambda_x(W) > 0$. Construct a sequence of sets W_n by setting $W_0 = W$ and

$$W_n = \Big\{T^{k_n(y)} y : y \in W_0 \text{ and } k_n(y) \text{ is minimal such that } \\ \pi_C(T^{k_n(y)}) = \pi_C(y) \text{ and } T^{k_n(y)} y \notin \bigcup_{k<n} W_k\Big\},$$

where the value of $k_n(y)$ is well-defined for each $y \in W_0$ and $n \in \mathbb{N}$, since all points in C_T return to their Voronoi domain infinitely often. Define a transformation $S_n : W_0 \to W_n$ as $S_n(y) = T^{k_n(y)} y$, and note that for all $n \in \mathbb{N}$ one has $\rho_{S_n}(y) \in \rho((\mathcal{V}_C)_c, (\mathcal{V}_C)_c)$, where, as earlier, ρ and ρ_{S_n} denote the cocycle maps. The region $\rho((\mathcal{V}_C)_c, (\mathcal{V}_C)_c)$ is precompact, since C is cocompact, and therefore using continuity of the Haar modulus $\Delta : G \to \mathbb{R}^{>0}$ one can pick $\epsilon > 0$ such that $\Delta(\rho_{S_n}(y)) > \epsilon$ for all $y \in W_0$ and all $n \in \mathbb{N}$.

Since S_n is composed of powers of T, Proposition 4.13 ensures that

$$\lambda_x(S_n W_0) = \int_{W_0} \Delta(\rho_{S_n}(y))\, d\lambda_x(y),$$

whence $\lambda_x(S_n W_0) \geq \epsilon\lambda_x(W_0)$ for each $n \in \mathbb{N}$. We now arrive at a contradiction, as W_n, $n \in \mathbb{N}$, form a pairwise disjoint infinite family of subsets of $(\mathcal{V}_C)_c$ whose measure is uniformly bounded away from zero by $\epsilon\lambda_x(W_0)$, which is impossible, since $\lambda_x((\mathcal{V}_C)_c) < \infty$ by cocompactness of C. This finishes the proof of the claim. $\square_{\text{claim}}$

We have established by now that $D_T \cap [x]_{\mathcal{R}_G} \subseteq D_x$ and, up to a null set, $C_T \cap [x]_{\mathcal{R}_G} \subseteq C_x$ by the claim above. Finally, $\mu(X \setminus (D_T \sqcup C_T)) = 0$ implies via Lemma 4.14 $\lambda_x((D_T \cap [x]_{\mathcal{R}_G}) \sqcup (C_T \cap [x]_{\mathcal{R}_G})) = 0$ for μ-almost all $x \in X$, and therefore

$$\lambda_x((D_T \cap [x]_{\mathcal{R}_G}) \triangle D_x) = 0 = \lambda_x((C_T \cap [x]_{\mathcal{R}_G}) \triangle C_x)$$

μ-almost surely. Sets D_T and C_T thus satisfy the conclusion of the proposition.

For the uniqueness part of the proposition, suppose D_T, C_T and D'_T, C'_T are two partitions of X such that

$$\lambda_x(D_T \triangle D_x) = 0 = \lambda_x(D'_T \triangle D_x) \text{ and } \lambda_x(C_T \triangle C_x) = 0 = \lambda_x(C'_T \triangle C_x)$$

for μ-almost all $x \in X$. One therefore also has $\forall^\mu x \in X\ \lambda_x(D_T \triangle D'_T) = 0 = \lambda_x(C_T \triangle C'_T)$, and hence $\mu(D_T \triangle D'_T) = 0$ and $\mu(C_T \triangle C'_T) = 0$ by Lemma 4.14. ■

We end this section with a natural definition which will be useful for analyzing elements of the full group.

Definition 4.17. Let $G \curvearrowright X$ be a free measure-preserving action of a locally compact Polish group on a standard probability space (X, μ), and let $T \in [G \curvearrowright X]$. Consider the T-invariant partition $X = D_T \sqcup C_T$ provided by the Hopf decomposition of T as per the previous proposition. We say that T is **dissipative** when $D_T = X$ and that T is **conservative** when $C_T = X$.

When G is discrete, observe that T is dissipative if and only if it is aperiodic (all its orbits are infinite), and that T is conservative if and only if it is periodic (all its orbits are finite).

Example 4.18. Let us give a general example of dissipative elements of the full group. Let $G \overset{\alpha}{\curvearrowright} X$ be a free measure-preserving action of a locally compact Polish group on a standard probability space (X, μ). If $g \in G$ generates a discrete infinite subgroup, then the element of the full group $\alpha(g)$ is dissipative. Indeed, the action of $\alpha(g)$ on each orbit is isomorphic to the g-action by left translation on G endowed with its right Haar measure, which is dissipative since it admits a Borel fundamental domain and has only infinite orbits. For instance, if $G = \mathbb{R}$, such a domain is given by the interval $[0, g)$ (or $(g, 0]$, if g is negative).

In Chapter 7, we build an interesting example of a conservative element in the full group of any free measure-preserving flow: its action on each orbit is actually ergodic, and its cocycle is bounded.

4.4 L^1 full groups and L^1 orbit equivalence

We now restrict ourselves to the setup where the acting group G is locally compact Polish and *compactly generated*, endowed with a maximal compatible norm $\|\cdot\|$ (the existence of such a norm for locally compact Polish group is equivalent to being compactly generated, see [52, Cor. 2.8 and Thm. 2.53]). For such a group, as explained in Section 2.2, it makes sense to talk about the associated L^1 full group by endowing the group with a maximal norm.

The following definition is the natural extension of the notion of L^1 orbit equivalence to the locally compact case, stated in terms of full groups.

Definition 4.19. Let α and β be the respective measure-preserving actions of two locally compact Polish compactly generated groups G and H on a standard probability space (X, μ). We say that α and β are L^1 **orbit equivalent** when there is a measure-preserving transformation $S \in \mathrm{Aut}(X, \mu)$ such that for all $g \in G$ and all $h \in H$,

$$S\alpha(g)S^{-1} \in [H \overset{\beta}{\curvearrowright} X]_1 \text{ and } S^{-1}\beta(h)S \in [G \overset{\alpha}{\curvearrowright} X]_1.$$

In other words, up to conjugating α by S, we have that the image of α is contained in the L^1 full group of β, and the image of β is contained in the L^1 full group of α.

We now show that L^1 full groups do remember actions up to L^1 orbit equivalence as abstract groups. This is done by finding a spatial realization of the isomorphism between the full groups. Such techniques originated in the work of H. Dye [15] and have been greatly generalized by D. H. Fremlin [18, 384D]. We recall that a subgroup G of $\mathrm{Aut}(X, \mu)$ is said to have **many involutions** if for any non-trivial measurable $A \subseteq X$ there exists a non-trivial involution $U \in G$ such that $\operatorname{supp} U \subseteq A$. The group of quasi-measure-preserving transformations of (X, μ) is denoted by $\mathrm{Aut}^*(X, \mu)$.

Theorem 4.20 (Fremlin). *Let G, H be subgroups of $\mathrm{Aut}(X, \mu)$ with many involutions. For any isomorphism $\psi : G \to H$ there exists $S \in \mathrm{Aut}^*(X, \mu)$ such that $\psi(T) = STS^{-1}$ for all $T \in G$.*

Proposition 4.21. *If the L^1 full groups of two ergodic measure-preserving actions of locally compact compactly generated Polish groups are isomorphic as abstract groups, then the two actions are L^1 orbit equivalent.*

Proof. Denote by $G \overset{\alpha}{\curvearrowright}$ and $H \overset{\beta}{\curvearrowright}$ the two actions on the same standard probability space (X, μ). Since the L^1 full groups of ergodic actions have many involutions (see, for example, Lemma 3.7), any isomorphism $\psi : [G \overset{\alpha}{\curvearrowright} X]_1 \to [H \overset{\beta}{\curvearrowright} X]_1$ admits a spatial realization by some $S \in \mathrm{Aut}^*(X, \mu)$. The Radon–Nikodym derivative of $S_*\mu$ with respect to μ is easily seen to be preserved by every element of $[H \overset{\beta}{\curvearrowright} X]_1$, and hence must be constant by ergodicity. We conclude that $S \in \mathrm{Aut}(X, \mu)$, and therefore by the definition the actions α and β are L^1 orbit equivalent. ■

Remark 4.22. Similarly to the finitely generated case [41, Sec. 8.1], one could define L^1 full orbit equivalence between actions as equality of L^1 full groups up to conjugacy, which is a strengthening of L^1 orbit equivalence (indeed the latter only requires inclusion of each acting group in the L^1 full group of the other acting group). It would be interesting to have examples of actions which are L^1 orbit equivalent, but not L^1 fully orbit equivalent.

We end this section by showing that L^1 orbit equivalence is equivalent to a stronger definition where we ask that, up to conjugating α by S, we moreover have that, on a full measure set $X_0 \subseteq X$, the α and β orbits coincide. This will be a direct consequence of the following proposition. The proof of this proposition is the same as that of [11, Prop. 3.8] which was not stated in the level of generality we need. Since it is short, we reproduce it here. We emphasize that when the acting groups are not discrete, the full measure set X_0 may very well fail to be α-invariant or β-invariant. In particular, when we say that the orbits coincide on X_0 we simply mean that they induce the same partition on X_0.

Proposition 4.23. *Let G and H be two locally compact Polish groups acting in a Borel measure-preserving manner on a standard probability space (X, μ), denote by α the G-action and suppose that $\alpha(G) \leq [H \curvearrowright X]$. Then there is a full measure Borel subset $X_0 \subseteq X$ such that*

$$\mathcal{R}_G \cap (X_0 \times X_0) \subseteq \mathcal{R}_H.$$

Proof. Let λ be the Haar measure on G. Since $\alpha(G) \leq [H \curvearrowright X]$, for all $g \in G$ and almost all $x \in X$, we have $gx \in Hx$. By Fubini's theorem, this implies that the Borel set

$$X_0 = \{x \in X : \text{ for } \lambda\text{-almost all } g \in G, \text{ we have } gx \in Hx\}$$

has full measure. Now let $x \in X_0$, and let $g_1 \in G$ be such that $g_1 x \in X_0$. We want to show that $g_1 x \in Hx$.

Since x and $g_1 x$ are in X_0, the sets

$$A = \{g \in G : gx \in Hx\} \quad \text{and} \quad B = \{g \in G : gx \in Hg_1 x\}$$

have full measure and so $A \cap B$ has full measure. Take $g \in A \cap B$, and note that $gx \in Hx \cap Hg_1 x$, so the two orbits Hx and $Hg_1 x$ intersect, hence $g_1 x \in Hx$. ■

Corollary 4.24. *Let G and H be compactly generated locally compact Polish groups, and let $\|\cdot\|_G$ and $\|\cdot\|_H$ be maximal norms on G and H, respectively. Two measure-preserving actions of G and H on a standard probability space (X, μ) are L^1 orbit equivalent if and only if they can be conjugated so as to share the same orbits on a full measure Borel subset $X_0 \subseteq X$, i.e., $\mathcal{R}_G \cap (X_0 \times X_0) = \mathcal{R}_H \cap (X_0 \times X_0)$, and there exist Borel maps $\gamma_G : G \times X_0 \to H$ and $\gamma_H : H \times X_0 \to G$ such that:*

(1) *for all $x \in X_0$, $g \cdot x = \gamma_G(g,x) \cdot x$ and $h \cdot x = \gamma_H(h,x) \cdot x$ whenever $gx \in X_0$ and $hx \in X_0$;*
(2) *$\int_{X_0} \|\gamma_G(g,x)\|_H \, d\mu(x) < +\infty$ and $\int_{X_0} \|\gamma_H(h,x)\|_G \, d\mu(x) < +\infty$ for all $g \in G$ and all $h \in H$.*

Proof. We may assume that the actions $G \curvearrowright X$ and $H \curvearrowright X$ are Borel. By the definition of L^1 full groups, the conditions stated in the corollary are sufficient to establish L^1 orbit equivalence. Conjugating the two actions, we may also assume that they share the same full group. Since the L^1 full groups contain the acting groups, we can apply Proposition 4.23 twice to obtain a full measure Borel subset $X_0 \subseteq X$ on which the orbits of the two actions coincide.

Let $\mathcal{F}(G)$ and $\mathcal{F}(H)$ denote the Effros Borel spaces associated with G and H, respectively. The orbit equivalence relations $\mathcal{R}_G$ and $\mathcal{R}_H$ are Borel, and consequently, the maps

$$G \times X_0 \ni (g,x) \mapsto \{h \in H : gx = hx\} \in \mathcal{F}(H),$$
$$H \times X_0 \ni (h,x) \mapsto \{g \in G : hx = gx\} \in \mathcal{F}(G)$$

are also Borel [6, Thm. 7.1.2]. Note that $\{h \in H : gx = hx\} \neq \varnothing$ whenever $gx \in X_0$, and similarly, $\{g \in G : hx = gx\} \neq \varnothing$ provided that $hx \in X_0$. By applying the Kuratowski–Ryll-Nardzewski selectors [31, 12.13], we can find Borel maps γ_G and γ_H that satisfy item (1) and the inequalities

$$\|\gamma_G(g,x)\|_H < D_H(x, gx) + 1, \quad \|\gamma_H(h,x)\|_G < D_G(x, hx) + 1,$$

where D_H and D_G are the metrics induced on the orbits by the respective actions. The integrability condition (2) now follows from the assumption that the actions have been conjugated to satisfy $G \leq [H \curvearrowright X]_1$ and $H \leq [G \curvearrowright X]_1$. ■

We will demonstrate in the final chapter that there exist free ergodic $\mathbb{R}$-flows that are not L^1 orbit equivalent. This result will be established by connecting L^1 orbit equivalence to flip Kakutani equivalence. In the discrete amenable setting, a key result due to T. Austin shows that entropy is preserved under L^1 orbit equivalence [3].

Question 4.25. *Let G be an amenable non-discrete non-compact compactly generated locally compact Polish group. Are there free measure-preserving ergodic actions of G which are not* L^1 *orbit equivalent?*

Chapter 5

Derived L^1 full groups for locally compact amenable groups

Given a measure-preserving action of a Polish normed group $(G, \|\cdot\|)$ on (X, μ), the **derived** L^1 **full group** $\mathfrak{D}([G \curvearrowright X]_1)$ of the action is, by definition, the (topological) derived subgroup of the L^1 full group $[G \curvearrowright X]_1$. Recall that this means that $\mathfrak{D}([G \curvearrowright X]_1)$ is the closure in $[G \curvearrowright X]_1$ of the subgroup generated by commutators, i.e., elements of the form $TUT^{-1}U^{-1}$, where $T, U \in [G \curvearrowright X]_1$. Provided the G-action is aperiodic, the latter can be described as a subgroup of $[G \curvearrowright X]_1$ in three different ways, using the fact that $[G \curvearrowright X]_1$ is *induction friendly*, as explained in Section 3.2 (see Corollary 3.16):

- $\mathfrak{D}([G \curvearrowright X]_1)$ is the closure of the group generated by involutions;
- $\mathfrak{D}([G \curvearrowright X]_1)$ is the closure of the group generated by 3-cycles;
- $\mathfrak{D}([G \curvearrowright X]_1)$ is the closure of the group generated by periodic elements.

In particular, all periodic elements of $[G \curvearrowright X]_1$ actually belong to $\mathfrak{D}([G \curvearrowright X]_1)$ (see Lemma 3.11 for the proof of this specific statement).

Compared to the previous chapter, we impose one further restriction on the acting group, and consider actions of a locally compact *amenable* Polish normed group $(G, \|\cdot\|)$. Appendix G of [7] contains an excellent review of amenability for both general topological groups and locally compact groups. As before, we fix a measure-preserving action $G \curvearrowright X$ on a standard probability space (X, μ), and let $D : \mathcal{R}_G \to \mathbb{R}^{\geq 0}$ denote the family of metrics induced onto the orbits by the norm. To ensure our results encompass both the non-compactly generated case and the situation in which the L^1 full group coincides with the entire full group of the action (as in [12]), we do not impose the condition that the norm be either proper or maximal. In particular, the norm may be bounded, which in turn implies that the metric D on the orbits is also bounded.

In Section 5.1, we construct a dense increasing chain of subgroups in $\mathfrak{D}([G \curvearrowright X]_1)$. This dense chain is utilized in the subsequent sections. In Section 5.2, we show that the amenability of the group G is reflected in the *whirly amenability* of $\mathfrak{D}([G \curvearrowright X]_1)$. Meanwhile, in Section 5.3, we prove, by a Baire category argument, that $\mathfrak{D}([G \curvearrowright X]_1)$ contains a dense 2-generated subgroup.

5.1 Dense chain of subgroups

An equivalence relation $\mathcal{R} \subseteq \mathcal{R}_G$ is said to be **uniformly bounded** if there is $M > 0$ and $X' \subseteq X$ such that $\mu(X \setminus X') = 0$ and $\sup_{(x_1,x_2)\in\mathcal{R}'} D(x_1, x_2) \leq M$, where $\mathcal{R}' = \mathcal{R} \cap X' \times X'$.

Lemma 5.1. *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space (X, μ). There exists a sequence of cross-sections C_n, $n \in \mathbb{N}$, and tessellations $\mathcal{W}_n$ over C_n such that for all $n \in \mathbb{N}$*

(1) $\mathcal{R}_{\mathcal{W}_n} \subseteq \mathcal{R}_{\mathcal{W}_{n+1}}$ *and* $\bigcup_{k\in\mathbb{N}} \mathcal{R}_{\mathcal{W}_k} = \mathcal{R}_G$ *(up to a null set);*

(2) $\mathcal{R}_{\mathcal{W}_n}$ *is uniformly bounded.*

Proof. Let C be a cocompact cross-section, $\mathcal{V}_C$ be the Voronoi tessellation over C, $\pi_{\mathcal{V}_C} : X \to C$ be the associated reduction, and $\nu = (\pi_{\mathcal{V}_C})_*\mu$ be the push-forward measure on C. Recall that $\mathcal{R}_{\mathcal{V}_C}$ is uniformly bounded since C is cocompact. Let E be the equivalence relation obtained by restricting $\mathcal{R}_G$ onto C. By a theorem of A. Connes, J. Feldman, and B. Weiss [13], E is hyperfinite on an invariant set of ν-full measure. Throwing away a G-invariant null set, we may write $E = \bigcup_n E_n$, where $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of Borel equivalence relations with finite classes. For $m, n \in \mathbb{N}$, define $A_{n,m}$ to be the set of points in the cross-section whose E_n-class is bounded in diameter by m:

$$A_{n,m} = \left\{c \in C : D(c_1, c_2) \leq m \text{ for all } c_1, c_2 \in C \text{ such that } c_1 E_n c \text{ and } c_2 E_n c\right\}.$$

Note that the sets $A_{n,m}$ are E_n-invariant, nested, and $\bigcup_m A_{n,m} = C$ for every $n \in \mathbb{N}$. Pick m_n so large as to ensure $\nu(C \setminus A_{n,m_n}) < 2^{-n}$ and let $B_n = \bigcap_{k\geq n} A_{k,m_k}$. The sets B_n are E_n-invariant, increasing, and $\lim_n \nu(B_n) = \nu(C)$. Define equivalence relations F_n on C by setting $c_1 F_n c_2$ whenever $c_1 = c_2$ or $c_1, c_2 \in B_n$ and $c_1 E_n c_2$. Note that $D(c_1, c_2) \leq m_n$ whenever $c_1 F_n c_2$. Let $C_n \subseteq C$ be a Borel transversal for F_n and define $\mathcal{W}_n = \{(c, x) \in C_n \times X : c F_n \pi_{\mathcal{V}_C}(x)\}$. It is straightforward to check that each $\mathcal{W}_n$ is a tessellation over C_n, and the equivalence relations $\mathcal{R}_{\mathcal{W}_n}$ satisfy the conclusions of the lemma. ∎

The equivalence relations $\mathcal{R}_{\mathcal{W}_n}$ produced by Lemma 5.1 give rise to a nested chain of groups $[\mathcal{R}_{\mathcal{W}_0}] \leq [\mathcal{R}_{\mathcal{W}_1}] \leq \cdots$. Some basic facts about such groups can be found in Appendix E.2. The following lemma establishes that such a chain is dense in the *derived* L^1 full group.

Lemma 5.2. *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space (X, μ) and let $(\mathcal{R}_n)_{n\in\mathbb{N}}$ be a sequence of equivalence relations as in Lemma 5.1. If the action is aperiodic, then the union $\bigcup_n [\mathcal{R}_n]$ is contained in the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$ and is dense therein.*

Proof. By definition, $[\mathcal{R}_n]$ is a subgroup of $[\mathcal{R}_G]$. Since equivalence relations $\mathcal{R}_n$ are uniformly bounded, we actually have $[\mathcal{R}_n] \leq [G \curvearrowright X]_1$, and the topology induced by the L^1 metric on $[\mathcal{R}_n]$ coincides with the topology induced from $[\mathcal{R}_G]$. Moreover, in view of Proposition D.7, $[\mathcal{R}_n]$ is topologically generated by periodic transformations, so we actually have $[\mathcal{R}_n] \leq \mathfrak{D}([G \curvearrowright X]_1)$ as a consequence of Lemma 3.11 and Corollary 3.16.

It remains to verify that the union $\bigcup_n [\mathcal{R}_n]$ is dense in $\mathfrak{D}([G \curvearrowright X]_1)$. To this end, recall that by Corollary 3.16, the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$ is topologically generated by involutions. So let $U \in \mathfrak{D}([G \curvearrowright X]_1)$ be an involution and set $X_n = \{x \in X : (x, U(x)) \in \mathcal{R}_n\}$, $n \in \mathbb{N}$. Note that X_n is U-invariant since U is an involution. Moreover, $\mu(X_n) \to 1$ as $\bigcup_n \mathcal{R}_n = \mathcal{R}_G$, and thus the induced transformations $U_{X_n} \in [\mathcal{R}_n]$ converge to U in the topology of $[G \curvearrowright X]_1$. We conclude that $\bigcup_n [\mathcal{R}_n]$ is dense in the derived L^1 full group. ∎

Corollary 5.3. *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space (X, μ). Suppose that almost every orbit of the action is uncountable. There exists a chain $H_0 \leq H_1 \leq \cdots \leq \mathfrak{D}([G \curvearrowright X]_1)$ of closed subgroups such that $\bigcup_n H_n$ is dense in $\mathfrak{D}([G \curvearrowright X]_1)$, and each H_n is isomorphic to $\mathrm{L}^0(Y_n, \nu_n, \mathrm{Aut}([0,1], \lambda))$ for some standard Lebesgue space (Y_n, ν_n). If, moreover, each orbit of the action has measure zero, then one can assume that all (Y_n, ν_n) are atomless and each H_n is isomorphic to $\mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))$.*

Proof. Apply Lemmas 5.1 and 5.2 to get a dense chain of subgroups $[\mathcal{R}_0] \leq [\mathcal{R}_1] \leq \cdots \leq \mathfrak{D}([G \curvearrowright X]_1)$ and use Corollary E.9 to deduce that each $[\mathcal{R}_n]$ has the desired form. ∎

Corollary 5.4. *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space (X, μ). If the action is aperiodic, then the set of periodic elements is dense in the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$.*

Proof. Consider a chain of subgroups $[\mathcal{R}_n]$ given by Lemma 5.2. Periodic elements are dense in these groups for their natural topology (see Proposition D.7 and the discussion preceding it). These topologies are compatible with the standard Borel structure of $\mathrm{Aut}(X, \mu)$ induced by the weak topology and therefore must refine the L^1 topology by the standard automatic continuity arguments [6, Sec. 1.6]. Hence, periodic elements are dense in all of $\mathfrak{D}([G \curvearrowright X]_1)$, as claimed. ∎

Corollary 5.4, together with Proposition 3.25, shows that the L^1 norm is maximal on derived L^1 full groups of aperiodic measure-preserving actions of locally compact amenable Polish normed groups (see Section 2.2 for a brief reminder on the maximality of norms). In particular, such groups are boundedly generated by [52, Thm. 2.53].

Theorem 5.5. *Let $(G, \|\cdot\|)$ be a locally compact amenable Polish normed group acting on a standard probability space (X, μ). If the action is aperiodic, then the L^1 norm is maximal on the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$.*

We do not know if the amenability hypothesis can be removed, even when G is discrete and the action is free.

5.2 Whirly amenability

Lemma 5.2 is a powerful tool to deduce various dynamical properties of derived L^1 full groups. Recall that a Polish group G is said to be **whirly amenable** if it is amenable and, for any continuous action of G on a compact space, any invariant measure is supported on the set of fixed points of the action. In particular, each such action has to have some fixed points, so whirly amenable groups are extremely amenable, meaning that all their continuous actions on compact spaces have fixed points.

Proposition 5.6. *Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space (X, μ). If μ is atomless, then the full group $[\mathcal{R}]$ is whirly amenable.*

Proof. In view of Proposition D.6, the full group $[\mathcal{R}]$ is isomorphic to

$$\mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))^{\epsilon_0} \times \mathrm{Aut}([0,1], \lambda)^{\kappa_0} \times \prod_{n \geq 1} \mathrm{L}^0([0,1], \lambda, \mathfrak{S}_n)^{\epsilon_0},$$

where $\mathfrak{S}_n$ is the group of permutations of an n-element set, and $\epsilon_n \in \{0, 1\}$, $\kappa_0 \leq \aleph_0$. Since a product of whirly amenable groups is whirly amenable, it suffices to show that the groups appearing in the decomposition above, namely $\mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))$, $\mathrm{Aut}([0,1], \lambda)$, and $\mathrm{L}^0([0,1], \lambda, \mathfrak{S}_n)$, $n \geq 1$, are whirly amenable.

The group $\mathrm{Aut}([0,1], \lambda)$ is whirly amenable by [22] (it is, in fact, a so-called Levy group). Finally, we apply a theorem of V. Pestov and F. M. Schneider [50], which asserts that a group $\mathrm{L}^0([0,1], \lambda, G)$ is whirly amenable if and only if G is amenable. This readily implies the whirly amenability of $\mathrm{L}^0([0,1], \lambda, \mathfrak{S}_n)$ and $\mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))$. ■

Remark 5.7. The assumption of μ being atomless cannot be omitted in the proposition above. Indeed, $[\mathcal{R}]$ will factor onto $\mathfrak{S}_n$ for some $n \geq 2$, as long as an $\mathcal{R}$-class contains at least 2 atoms of μ of the same measure. However, if all μ-atoms within each $\mathcal{R}$-class have distinct measures, then the restriction of $[\mathcal{R}]$ onto the atomic part of X is trivial, which suffices to conclude the whirly amenability of the group $[\mathcal{R}]$.

Theorem 5.8. *Let $G \curvearrowright X$ be a measure-preserving action of an amenable locally compact Polish normed group on a standard probability space (X, μ). If the action is aperiodic, then the derived L^1 full group $\mathfrak{D}([G \curvearrowright X]_1)$ is whirly amenable. In particular, $[G \curvearrowright X]_1$ is amenable.*

Proof. Lemma 5.2 shows that $\mathfrak{D}([G \curvearrowright X]_1)$ has an increasing dense chain of subgroups H_n of the form $[\mathcal{R}_n]$, where $\mathcal{R}_n$ are smooth measurable equivalence relations on X. Proposition 5.6 applies and shows that the groups H_n are whirly amenable. The latter is sufficient to conclude the whirly amenability of $\mathfrak{D}([G \curvearrowright X]_1)$, as any invariant measure for the action of the derived group is also invariant for the induced H_n-actions. Hence, it has to be supported on the intersection of the fixed points of all H_n, which coincides with the set of fixed points for the action of $\mathfrak{D}([G \curvearrowright X]_1)$.

The fact that $[G \curvearrowright X]_1$ is amenable now follows from the fact that every abelian group is amenable and that every amenable extension of an amenable group must itself be amenable (for instance, see [7, Prop. G.2.2]). ∎

Remark 5.9. Note that, in general, $[G \curvearrowright X]_1$ is not extremely amenable. For flows, it factors onto $\mathbb{R}$ via the index map (see Chapter 6). Since $\mathbb{R}$ admits continuous actions on compact spaces without fixed points, $[\mathbb{R} \curvearrowright X]_1$ is not extremely amenable (and in particular, it is not whirly amenable) for any free measure-preserving flow.

Corollary 5.10. *Let $G \curvearrowright X$ be a free measure-preserving action of a unimodular locally compact Polish group on a standard probability space (X, μ). The following are equivalent:*

(1) *G is amenable.*

(2) *$[G \curvearrowright X]_1$ is amenable.*

(3) *The derived* L^1 *full group $\mathfrak{D}([G \curvearrowright X]_1)$ is amenable.*

(4) *The derived* L^1 *full group $\mathfrak{D}([G \curvearrowright X]_1)$ is extremely amenable.*

(5) *The derived* L^1 *full group $\mathfrak{D}([G \curvearrowright X]_1)$ is whirly amenable.*

Proof. We established the implication (1) $\Longrightarrow$ (5) in Theorem 5.8. The chain of implications (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) is straightforward, and (3) $\Longrightarrow$ (2) follows from the stability of amenability under group extensions, which was already discussed in Theorem 5.8.

For the last implication (2) $\Longrightarrow$ (1), first recall that the orbit full group of the action is generated by involutions. It follows that the orbit full group is topologically generated by involutions whose cocycles are integrable (actually, one can even ask that the cocycles are bounded). In particular, the L^1 full group $[G \curvearrowright X]_1$ is dense in the orbit full group, and so, assuming (2), we conclude that the orbit full group $[G \curvearrowright X]$ is amenable. The amenability of G then follows from [12, Thm. 5.1]. ∎

Remark 5.11. We have to require unimodularity to apply [12, Thm. 5.1]. It seems likely that the unimodularity hypothesis can be dropped in this result, but we do not pursue this question further.

5.3 Topological generators

We now concern ourselves with the question of determining the topological rank of derived L^1 full groups. Our approach will be based on the dense chain of subgroups established in Corollary 5.3, and the first step is to study the topological rank of the group $\mathrm{L}^0([0,1], \mathrm{Aut}([0,1]))$.

Let (Y, ν) and (Z, λ) be standard Lebesgue spaces. Consider the product space $Y \times Z$ equipped with the product measure $\nu \times \lambda$ and let $\mathcal{R}$ be the product of the discrete equivalence relation on Y and the anti-discrete on Z; in other words, $(y_1, z_1)\mathcal{R}(y_2, z_2)$ if and only if $y_1 = y_2$. As discussed in Appendix D, the following two groups are one and the same:

(1) the full group $[\mathcal{R}]$;

(2) the topological group $\mathrm{L}^0(Y, \nu, \mathrm{Aut}(Z, \lambda))$.

In particular, we may and do endow $[\mathcal{R}]$ with the Polish group topology induced by its natural identification with $\mathrm{L}^0(Y, \nu, \mathrm{Aut}(Z, \lambda))$.

Suppose additionally that (Z, λ) is atomless. Pick a hyperfinite ergodic measure-preserving equivalence relation E on (Z, λ). We claim that $\mathrm{APER}(Z) \cap [E]$ is dense in $\mathrm{Aut}(Z, \lambda)$, where $\mathrm{APER}(Z)$ stands for the collection of aperiodic automorphisms of Z. Indeed, first note that by [32, Prop. 3.1], the full group $[E]$ is weakly dense in $\mathrm{Aut}(Z, \lambda)$. Let us then pick any aperiodic $T \in [E]$. It follows from [32, Thm. 2.4] that the $\mathrm{Aut}(Z, \lambda)$-conjugacy class of T is weakly dense in $\mathrm{Aut}(X, \lambda)$. By the continuity of the conjugacy action and weak density, the $[E]$-conjugacy class of T is weakly dense as well, which proves our claim since this conjugacy class is clearly contained in $\mathrm{APER}(Z) \cap [E]$.

Now set $\mathcal{R}_0 = \mathrm{id}_Y \times E$ to be the equivalence relation on $Y \times Z$ given by the condition $(y_1, z_1)\mathcal{R}_0(y_2, z_2)$ whenever $y_1 = y_2$ and $z_1 E z_2$. A standard application of the Jankov–von Neumann uniformization theorem yields the following lemma.

Lemma 5.12. $\mathrm{APER}(Y \times Z) \cap [\mathcal{R}_0]$ *is dense in* $[\mathcal{R}] \simeq \mathrm{L}^0(Y, \nu, \mathrm{Aut}([0,1], \lambda))$.

Our first goal is to establish that the topological rank of $[\mathcal{R}]$ is 2. We do so by first verifying this under the assumption that (Y, ν) is atomless and then deducing the general case.

We say that a Polish group G is **generically k-generated**, where $k \in \mathbb{N}$, if the set of k-tuples $(g_1, \dots, g_k) \in G^k$ that generate a dense subgroup of G is dense in G^k. Note that the set of such tuples is always a G_δ set, so if G is generically k-generated, then a comeager set of k-tuples generates a dense subgroup of G.

Proposition 5.13. *Suppose that* (Y, ν) *is atomless. The Polish group* $[\mathcal{R}]$ *is generically 2-generated.*

Proof. By [39, Thm 5.1], the set of pairs

$$(S,T) \in (\mathrm{APER}(Y \times Z) \cap [\mathcal{R}_0]) \times [\mathcal{R}_0]$$

such that $\overline{\langle S,T\rangle} = [\mathcal{R}_0]$ is dense G_δ for the uniform topology. In view of Lemma 5.12, this implies that $[\mathcal{R}]$ is generically 2-generated. ∎

Lemma 5.14. *For all Polish groups G and H, one has*

$$\mathrm{rk}(G \times H) \geq \max\{\mathrm{rk}(G), \mathrm{rk}(H)\}.$$

If $G \times H$ is generically k-generated, then so are G and H as well.

Proof. The inequality on ranks is immediate from the trivial observation that if $\langle (g_1, h_1), \dots, (g_k, h_k)\rangle$ is dense in $G \times H$, then $\langle g_1, \dots, g_k\rangle$ is dense in G and $\langle h_1, \dots, h_k\rangle$ is dense in H.

Suppose $G \times H$ is generically k-generated. Pick an open set $U \subseteq G^k$ and note that $U \times H^k$ corresponds to an open subset of $(G \times H)^k$ via the isomorphism $(G \times H)^k \simeq G^k \times H^k$. Since $G \times H$ is generically k-generated, there is a tuple $(g_i, h_i)_{i=1}^k \in (G \times H)^k$ that generates a dense subgroup and $(g_i, h_i)_{i=1}^k \in U \times H^k$. We conclude that $(g_i)_{i=1}^k \in U$ generates a dense subgroup of G, and the lemma follows. ∎

Lemma 5.15. *For any Polish group G*

$$\mathrm{rk}(\mathrm{L}^0([0,1], \lambda, G)) = \mathrm{rk}\big(\mathrm{L}^0([0,1], \lambda, G) \times G^{\mathbb{N}}\big).$$

If $\mathrm{L}^0([0,1], \lambda, G)$ is generically k-generated for some $k \in \mathbb{N}$, then so is the group $\mathrm{L}^0([0,1], \lambda, G) \times G^{\mathbb{N}}$.

Proof. In view of Lemma 5.14, $\mathrm{rk}(\mathrm{L}^0([0,1], \lambda, G)) \leq \mathrm{rk}\big(\mathrm{L}^0([0,1], \lambda, G) \times G^{\mathbb{N}}\big)$, and since the group G is separable, we only need to consider the case when the rank $\mathrm{rk}(\mathrm{L}^0([0,1], \lambda, G))$ is finite.

It is notationally convenient to shrink the interval and work with the group

$$\mathrm{L}^0([0,1/2], \lambda, G) \times G^{\mathbb{N}}$$

instead, as it can naturally be viewed as a closed subgroup of $\mathrm{L}^0([0,1], \lambda, G)$ via the identification $f \times (g_i)_{i\in\mathbb{N}} \mapsto \zeta$, where

$$\zeta(t) = \begin{cases} f(t) & \text{if } 0 \leq t < 1/2, \\ g_i & \text{if } 1 - 2^{-i-1} \leq t < 1 - 2^{-i-2} \text{ for } i \in \mathbb{N}. \end{cases}$$

Pick families $(\xi_l)_{l\in\mathbb{N}}$ dense in $\mathrm{L}^0([0,1/2], \lambda, G)$, and $(h_m)_{m\in\mathbb{N}}$ dense in G.

Let us call a function $\alpha : \mathbb{N} \to \mathbb{N}$ a multi-index if $\alpha(i) = 0$ for all but finitely many $i \in \mathbb{N}$. We use $\mathbb{N}^{<\mathbb{N}}$ to denote the set of all multi-indices. Given $\alpha \in \mathbb{N}^{<\mathbb{N}}$, we define $h_\alpha = (h_{\alpha(i)})_{i\in\mathbb{N}} \in G^{\mathbb{N}}$. Note that $\{h_\alpha : \alpha \in \mathbb{N}^{<\mathbb{N}}\}$ is dense in $G^{\mathbb{N}}$, and thus $\{\xi_l \times h_\alpha : l \in \mathbb{N}, \alpha \in \mathbb{N}^{<\mathbb{N}}\}$ is a dense family in $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$.

Pick a tuple $f_1,\dots,f_k \in \mathrm{L}^0([0,1],\lambda,G)$ that generates a dense subgroup. For each pair $(l,\alpha)\in\mathbb{N}\times\mathbb{N}^{<\mathbb{N}}$, there exists a sequence of reduced words $(w_n^{l,\alpha})_{n\in\mathbb{N}}$ in the free group on k generators such that $w_n^{l,\alpha}(f_1,\dots,f_k)$ converges to $\xi_l\times h_\alpha$ in measure. By passing to a subsequence, we may assume that $w_n^{l,\alpha}(f_1,\dots,f_k)\to\xi_l\times h_\alpha$ pointwise almost surely. In other words, the set

$$P_{l,\alpha} = \left\{t\in[0,1] : w_n^{l,\alpha}(f_1,\dots,f_k)(t)\to(\xi_l\times h_\alpha)(t)\right\}$$

has Lebesgue measure 1 for each $(l,\alpha)\in\mathbb{N}\times\mathbb{N}^{<\mathbb{N}}$, and hence so does the set

$$P = \bigcap_{l\in\mathbb{N}}\ \bigcap_{\alpha\in\mathbb{N}^{<\mathbb{N}}} P_{l,\alpha}.$$

Pick some $t_j \in P\cap[1-2^{-j-1},1-2^{-j-2})$, $j\in\mathbb{N}$, and set

$$\tilde{f}_i(t) = \begin{cases} f_i(t) & \text{for } 0\le t<1/2,\\ f_i(t_j) & \text{for } 1-2^{-j-1}\le t<1-2^{-j-2} \text{ for } j\in\mathbb{N}.\end{cases}$$

Elements $\tilde{f}_i$ naturally belong to $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, and we claim that they generate a dense subgroup therein, witnessing $\mathrm{rk}(\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}})\le k$. To this end, recall that $w_n^{l,\alpha}(f_1,\dots,f_k)\to\xi_l\times h_\alpha$ pointwise almost surely. In particular,

$$w_n^{l,\alpha}(f_1,\dots,f_k)\restriction_{[0,1/2]}\to\xi_l\times h_\alpha\restriction_{[0,1/2]}$$

in measure and, for each $j\in\mathbb{N}$,

$$w_n^{l,\alpha}(f_1,\dots,f_k)(t_j)\to(\xi_l\times h_\alpha)(t_j)=h_{\alpha(j)}$$

is guaranteed by choosing $t_j\in P$. We conclude that

$$w_n^{l,\alpha}(\tilde{f}_1,\dots,\tilde{f}_k)\to\xi_l\times h_\alpha$$

in $\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$, and therefore

$$\mathrm{rk}(\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}})\le k.$$

Finally, suppose that $\mathrm{L}^0([0,1],\lambda,G)$ is generically k-generated. Choose open sets $U_i\subseteq\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}}$,$1\le i\le k$. Shrinking them if necessary, we may assume that all U_i have the form $U_i = A_0^i\times A_1^i\times\cdots\times A_n^i\times G^{\mathbb{N}}$, where A_0^i is open in $\mathrm{L}^0([0,1/2],\lambda,G)$, and A_j^i, $j\ge1$, are open in G.

Pick $V_i\subseteq\mathrm{L}^0([0,1],\lambda,G)$, $1\le i\le k$, to consist of those functions f satisfying $f|_{[0,1/2]}\in A_0$ and $f(t)\in A_j$ for all $t\in[1-2^{-j-1},1-2^{-j-2})$, $1\le j\le n$. Note that $V_i\cap\mathrm{L}^0([0,1/2],\lambda,G)\times G^{\mathbb{N}} = U_i$.

Since $\mathrm{L}^0([0,1],\lambda,G)$ is assumed to be generically k-generated, there is a tuple $(f_1,\ldots,f_k)$ generating a dense subgroup in $\mathrm{L}^0([0,1],\lambda,G)$ such that $f_i \in V_i$ for each i. Running the above construction, we get a tuple

$$(\tilde{f}_1,\ldots,\tilde{f}_k) \in \mathrm{L}^0([0,1/2],\lambda,G) \times G^{\mathbb{N}}$$

such that $\tilde{f}_i \in U_i$, $1 \le i \le k$, whence $\mathrm{L}^0([0,1/2],\lambda,G) \times G^{\mathbb{N}}$ is generically k-generated. ∎

Lemma 5.15 remains valid if we take the product with a finite power of G, which follows from Lemma 5.14.

Corollary 5.16. *For any Polish group G and any $m \in \mathbb{N}$, one has*

$$\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G)) = \mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G)) \times G^m.$$

If $\mathrm{rk}(\mathrm{L}^0([0,1],\lambda,G))$ *is generically k-generated for some $k \in \mathbb{N}$, then so is the group* $\mathrm{L}^0([0,1],\lambda,G) \times G^m$.

We may now strengthen Proposition 5.13 by dropping the assumption on (Y,ν) being atomless.

Proposition 5.17. *Let (Y,ν) be a standard Lebesgue space and (Z,λ) be a standard probability space. The Polish group* $\mathrm{L}^0(Y,\nu,\mathrm{Aut}(Z,\lambda))$ *is generically* 2*-generated.*

Proof. Let Y_a be the set of atoms of Y, put $Y_0 = Y \setminus Y_a$ and $\nu_0 = \nu\restriction_{Y_0}$. The group $\mathrm{L}^0(Y,\nu,\mathrm{Aut}(Z,\lambda))$ is naturally isomorphic to

$$\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}(Z,\lambda)) \times \mathrm{Aut}(Z,\lambda)^{|Y_a|}.$$

An application of Proposition 5.13 together with Lemma 5.15 or Corollary 5.16 (depending on whether Y_a is infinite or not) finishes the proof. ∎

Proposition 5.18. *Let G be a Polish group and let $H_0 \le H_1 \le \cdots \le G$ be a dense chain of Polish subgroups, $\overline{\bigcup_n H_n} = G$. If each H_n is generically k-generated, then G is generically k-generated.*

Proof. We need to show that for any open $U \subseteq G^k$ and any open $V \subseteq G$ there is a tuple $(g_1,\ldots,g_k) \in U$ such that $\langle g_1,\ldots,g_k\rangle \cap V \neq \varnothing$. Since groups H_n are nested and $\bigcup_n H_n$ is dense in G, there is n so large that $U \cap H_n^k \neq \varnothing$ and $V \cap H_n \neq \varnothing$. It remains to use the fact that H_n is generically k-generated to find the required tuple. ∎

Theorem 5.19. *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact amenable Polish normed group on a standard probability space (X,μ). If almost every orbit of the action is uncountable, then the derived* L^1 *full group* $\mathfrak{D}([G \curvearrowright X]_1)$ *is generically* 2*-generated and has topological rank* 2.

Proof. In view of Corollary 5.3, there is a chain of subgroups

$$H_0 \leq H_1 \leq \cdots \leq \mathfrak{D}([G \curvearrowright X]_1), \quad \overline{\bigcup_n H_n} = \mathfrak{D}([G \curvearrowright X]_1),$$

where each H_n is isomorphic to $\mathrm{L}^0(Y_n, \nu_n, \mathrm{Aut}([0,1], \lambda))$ for some standard Lebesgue space (Y_n, ν_n). By Proposition 5.17, every H_n is generically 2-generated, and we may apply Proposition 5.18 to conclude that $\mathfrak{D}([G \curvearrowright X]_1)$ is generically 2-generated. In particular, its topological rank is at most 2. To see that its topological rank is actually equal to 2, simply note that $\mathfrak{D}([G \curvearrowright X]_1)$ is not abelian (e.g., by the proof of Proposition 3.9). ∎

The assumption for orbits to be uncountable is essential, and Theorem 5.19 is in striking contrast to the dynamical interpretation of the topological rank of derived L^1 full groups for actions of discrete groups. As shown in [41, Thm. 4.3], given an aperiodic measure-preserving action of a finitely generated group $\Gamma \curvearrowright X$, the topological rank of $\mathfrak{D}([\Gamma \curvearrowright X]_1)$ is finite if and only if the action has finite Rokhlin entropy.

Chapter 6

The index map for L^1 full groups of flows

We now turn our attention to **flows**, i.e., measure-preserving actions of $\mathbb{R}$. Since the group of reals is locally compact, amenable, unimodular, and, of course, Polish, all of the results in the previous chapters apply to $\mathbb{R}$-flows. A much more in-depth understanding of L^1 full groups of flows is possible and is based on the existence of the so-called *index map*, which we define and investigate in this chapter. This map is a continuous homomorphism from the L^1 full group of the flow to the additive group of reals, which can be thought of as measuring the average shift distance. When the flow is ergodic, such averages are the same across orbits. By taking the ergodic decomposition of the flow $\mathcal{F}$, we can adopt a slightly more general vantage point and view the index map $\mathcal{I}$ as a homomorphism into the L^1 space of functions on the space of invariant measures $(\mathcal{E}, p)$, $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$.

Understanding the kernel of the index map is a task of fundamental importance. We will subsequently identify $\ker \mathcal{I}$ with the topological derived subgroup of $[\mathcal{F}]_1$ (Theorem 10.1). This will allow us to describe the abelianizations of L^1 full groups of flows and estimate the number of their topological generators.

It has already been mentioned that any element T of a full group of a flow induces Lebesgue measure-preserving transformations on orbits (Section 4.2). When T furthermore belongs to the L^1 full group, these transformations are special—they leave "half-lines" invariant up to a set of finite measure. Such transformations form the so-called **commensurating group**. Let us therefore begin with a more formal treatment of this group, which has already appeared in the literature before, for instance in [51].

6.1 Self-commensurating automorphisms of a subset

Consider an infinite measure space (Z, λ). We say that two measurable sets $A, B \subseteq Z$ are **commensurate** if the measure of their symmetric difference is finite, $\lambda(A \triangle B) < \infty$. The relation of being commensurate is an equivalence relation, and all sets of finite measure fall into a single class. Note also that if A and B are both commensurate to some C, then so is the intersection $A \cap B$; in other words, all equivalence classes of commensurability are closed under finite intersections.

Let $\mathfrak{C}(B)$ denote the set of all measurable $A \subseteq Z$ that are commensurate to B. Fix some $Y \subseteq Z$ and consider the semigroup of measure-preserving transformations between elements of $\mathfrak{C}(Y)$. More precisely, let $\mathrm{Iso}^\star(Y, \lambda)$ be the set of measure-preserving partial bijections $T : A \to B$ between sets $A, B \in \mathfrak{C}(Y)$, which we call the **self-commensurating semigroup** of (Y, λ).

Recall that we denote the domain of T as $\operatorname{dom} T$ and its range as $\operatorname{rng} T$. For partial transformations $S : A \to B$ and $T : A' \to B'$, the composition $T \circ S$ has a domain given by $A \cap S^{-1}(A')$. As always, we identify two maps if they differ only on a null set. Since the classes of commensurability are closed under finite intersections, the set $\operatorname{Iso}^\star(Y, \lambda)$ forms a semigroup with respect to composition.

This semigroup carries a natural equivalence relation: $T \sim S$ whenever the transformations disagree on a set of finite measure, $\lambda(\{x : Tx \neq Sx\}) < \infty$. This equivalence is, moreover, a congruence, i.e., if $T_1 \sim S_1$ and $T_2 \sim S_2$, then $T_1 \circ T_2 \sim S_1 \circ S_2$. One may therefore push the semigroup structure from $\operatorname{Iso}^\star(Y, \lambda)$ onto the set of equivalence classes, which we denote by $\operatorname{Aut}^\star(Y, \lambda)$. An important observation is that $\operatorname{Aut}^\star(Y, \lambda)$ is a group. Indeed, the identity corresponds to the map $x \mapsto x$ on Y, and for a representative $T \in \operatorname{Iso}^\star(Y, \lambda)$, its inverse inside $\operatorname{Aut}^\star(Y, \lambda)$ is, naturally, given by $T^{-1} : \operatorname{rng} T \to \operatorname{dom} T$. We call $\operatorname{Aut}^\star(Y, \lambda)$ the **self-commensurating automorphism group** of Y.

The self-commensurating semigroup admits an important homomorphism into the reals, $\mathcal{I} : \operatorname{Iso}^\star(Y, \lambda) \to \mathbb{R}$, called the **index map** and defined by

$$\mathcal{I}(T) = \lambda(\operatorname{dom} T \setminus \operatorname{rng} T) - \lambda(\operatorname{rng} T \setminus \operatorname{dom} T).$$

Lemma 6.1. *For all $T \in \operatorname{Iso}^\star(Y, \lambda)$, the index map satisfies the following:*

(1) *if $A \in \mathfrak{C}(Y)$ is such that $\operatorname{dom} T \subseteq A$ and $\operatorname{rng} T \subseteq A$, then*

$$\mathcal{I}(T) = \lambda(A \setminus \operatorname{rng} T) - \lambda(A \setminus \operatorname{dom} T);$$

(2) *if $T' \in \operatorname{Iso}^\star(Y, \lambda)$ is a restriction of T', that is $T' = T \restriction_{\operatorname{dom} T'}$, then $\mathcal{I}(T') = \mathcal{I}(T)$.*

Proof. (1) If $A \subseteq Z$ is commensurate to Y and $\operatorname{dom} T \subseteq A$, $\operatorname{rng} T \subseteq A$, then

$$\begin{aligned}\mathcal{I}(T) &= \lambda(\operatorname{dom} T \setminus \operatorname{rng} T) - \lambda(\operatorname{rng} T \setminus \operatorname{dom} T) \\ &= \lambda(A \setminus \operatorname{rng} T) - \lambda(A \setminus (\operatorname{dom} T \cup \operatorname{rng} T)) \\ &\qquad - (\lambda(A \setminus \operatorname{dom} T) - \lambda(A \setminus (\operatorname{dom} T \cup \operatorname{rng} T))) \\ &= \lambda(A \setminus \operatorname{rng} T) - \lambda(A \setminus \operatorname{dom} T).\end{aligned}$$

(2) If $T' \in \operatorname{Iso}^\star(Y, \lambda)$ is a restriction of T, then

$$T(\operatorname{dom} T \setminus \operatorname{dom} T') = \operatorname{rng} T \setminus \operatorname{rng} T'.$$

Thus, for any $A \in \mathfrak{C}(Y)$ containing both $\operatorname{dom} T$ and $\operatorname{rng} T$, item (1) implies

$$\begin{aligned}\mathcal{I}(T) &= \lambda(A \setminus \operatorname{dom} T) - \lambda(A \setminus \operatorname{rng} T) \\ &= \lambda(A \setminus \operatorname{dom} T') - \lambda(\operatorname{dom} T \setminus \operatorname{dom} T') - (\lambda(B \setminus \operatorname{rng} T') - \lambda(\operatorname{rng} T \setminus \operatorname{rng} T')) \\ &= \lambda(A \setminus \operatorname{dom} T') - \lambda(A \setminus \operatorname{rng} T') = \mathcal{I}(T'),\end{aligned}$$

where the equality $\lambda(\operatorname{dom} T \setminus \operatorname{dom} T') = \lambda(\operatorname{rng} T \setminus \operatorname{rng} T')$ is based on T being measure-preserving. ∎

Proposition 6.2. *The index map* $\mathcal{I} : \mathrm{Iso}^\star(Y, \lambda) \to \mathbb{R}$ *is a homomorphism. Moreover, if* $T, S \in \mathrm{Iso}^\star(Y, \lambda)$ *are equivalent,* $T \sim S$*, then* $\mathcal{I}(T) = \mathcal{I}(S)$.

Proof. In view of Lemma 6.1(2), to check that $\mathcal{I}(T_1 \circ T_2) = \mathcal{I}(T_1) + \mathcal{I}(T_2)$, we may pass to restrictions of these transformations and assume that $\operatorname{rng} T_2 = \operatorname{dom} T_1$. Pick a set $A \in \mathfrak{C}(Y)$ large enough to contain the domains and ranges of T_1 and T_2. By Lemma 6.1(1),

$$\begin{aligned}\mathcal{I}(T_1 \circ T_2) &= \lambda(A \setminus \operatorname{rng} T_1) - \lambda(A \setminus \operatorname{dom} T_2)\\ &= \lambda(A \setminus \operatorname{rng} T_1) - \lambda(A \setminus \operatorname{dom} T_1) + \lambda(A \setminus \operatorname{rng} T_2) - \lambda(A \setminus \operatorname{dom} T_2)\\ &= \mathcal{I}(T_1) + \mathcal{I}(T_2).\end{aligned}$$

For the moreover part, suppose that $T, S \in \mathrm{Iso}^\star_Y(Y, \lambda)$ are equivalent. Let U be the restriction of T and S onto the set $\{x : Tx = Sx\}$. Using Lemma 6.1(2) once again, we get $\mathcal{I}(T) = \mathcal{I}(U) = \mathcal{I}(S)$. Hence, the index map is invariant under the equivalence relation $\sim$. ∎

The proposition above implies that the index map respects the relation $\sim$, and hence gives rise to a map from $\mathrm{Aut}^\star(Y, \lambda)$ to the reals.

Corollary 6.3. *The index map factors to a group homomorphism*

$$\mathcal{I} : \mathrm{Aut}^\star(Y, \lambda) \to \mathbb{R}.$$

6.2 The commensurating automorphism group

Let us again consider an infinite measure space (Z, λ) and $Y \subseteq Z$ a measurable subset. We now define the **commensurating automorphism group of** Y **in** Z as the group of all measure-preserving transformations $T \in \mathrm{Aut}(Z, \lambda)$ such that $\lambda(Y \triangle T(Y)) < \infty$. We denote this group by $\mathrm{Aut}_Y(Z, \lambda)$.

Every $T \in \mathrm{Aut}_Y(Z, \lambda)$ naturally gives rise to an element of $\mathrm{Aut}^\star(Y, \lambda)$ by considering its restriction $T \restriction_Y$. The following lemma shows that in this case we may use any other set A commensurate to Y instead without changing the corresponding element of the commensurating group.

Lemma 6.4. *Let* $T \in \mathrm{Aut}(Z, \lambda)$ *be a measure-preserving automorphism. If* $T \restriction_A \in \mathrm{Iso}^\star(Y, \lambda)$ *for some* $A \in \mathfrak{C}(Y)$*, then* $T \restriction_B \in \mathrm{Iso}^\star(Y, \lambda)$ *and* $T \restriction_B \sim T \restriction_A$ *for all* $B \in \mathfrak{C}(Y)$.

Proof. Since commensuration is an equivalence relation and A is commensurate to Y, the assumption $T \restriction_A \in \mathrm{Iso}^\star(Y, \lambda)$ is equivalent to $\lambda(A \triangle T(A)) < \infty$. Moreover, given $B \in \mathfrak{C}(Y)$, we only need to show that $\lambda(B \triangle T(B))$ is finite in order to conclude that

$T \restriction_B \in \mathrm{Iso}^\star(Y,\lambda)$. So we compute

$$\begin{aligned}\lambda(B \triangle T(B)) =& \lambda(B \setminus T(B)) + \lambda(T(B) \setminus B)\\ \le& \lambda(A \setminus T(A)) + \lambda(B \setminus A) + \lambda(T(A \setminus B))\\ &\quad + \lambda(T(A) \setminus A) + \lambda(A \setminus B) + \lambda(T(B \setminus A))\\ =& \lambda(A \triangle T(A)) + 2\lambda(A \triangle B) < \infty.\end{aligned}$$

Thus, the measure $\lambda(B \triangle T(B))$ is finite. Hence $T \restriction_B \in \mathrm{Iso}^\star(Y,\lambda)$ for all $B \in \mathfrak{C}(Y)$. Finally, $T \restriction_A \sim T \restriction_B$, since these transformations agree on $A \cap B$. ■

To summarize, if $T \restriction_A \in \mathrm{Iso}^\star(Y,\lambda)$ for some $A \in \mathfrak{C}(Y)$, then all restrictions $T \restriction_B$, $B \in \mathfrak{C}(Y)$, are pairwise equivalent. Hence, they correspond to the same element $T \restriction_Y \in \mathrm{Aut}^\star(Y,\lambda)$. According to Proposition 6.2, the index $\mathcal{I}(T \restriction_Y)$ of this element can be computed as $\mathcal{I}(T \restriction_Y) = \lambda(B \setminus T(B)) - \lambda(B \setminus T^{-1}(B))$ for any $B \in \mathfrak{C}(Y)$.

6.3 Index map on L^1 full groups of $\mathbb{R}$-flows

Let $\mathcal{F} = \mathbb{R} \curvearrowright X$ be a free measure-preserving Borel flow, let $[\mathcal{F}]_1$ be the associated L^1 full group, where we endow $\mathbb{R}$ with the standard Euclidean norm, and let $T \in [\mathcal{F}]_1$. The action of $r \in \mathbb{R}$ upon $x \in X$ is denoted additively by $x + r$. Recall that the cocycle of T is denoted by $\rho_T : X \to \mathbb{R}$ and is defined by the equality $T(x) = x + \rho_T(x)$ for all $x \in X$. We are going to argue that, on every orbit, T induces a measure-preserving transformation that belongs to the commensurate group of $\mathbb{R}^{\ge 0}$, when the orbit is identified with the real line.

Consider the function $f : \mathcal{R}_{\mathcal{F}} \to \{-1, 0, 1\}$ defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x < y < T(x),\\ -1 & \text{if } T(x) < y < x,\\ 0 & \text{otherwise.}\end{cases}$$

One can think of f as a "charge function" that spreads charge $+1$ over each interval $(x, T(x))$ and -1 over $(T(x), x)$. Note that we have both

$$\int_{\mathbb{R}} f(x, x+r)\, d\lambda(r) = \rho_T(x) \quad \text{and} \quad \int_{\mathbb{R}} |f(x, x+r)|\, d\lambda(r) = |\rho_T(x)|\,.$$

Since T belongs to the L^1 full group, its cocycle is integrable, which means that f is M-integrable (see Section 4.2). By the mass-transport principle, the following integrals are equal and finite:

$$\int_X \int_{\mathbb{R}} |f(x, x+r)|\, d\lambda(r) d\mu(x) = \int_X \int_{\mathbb{R}} |f(x+r, x)|\, d\lambda(r) d\mu(x).$$

In particular, the integral $\int_{\mathbb{R}} |f(x+r,x)|\, d\lambda(r)$ is finite for almost all x.

Let $T_x \in \mathrm{Aut}(\mathbb{R},\lambda)$ denote the transformation induced by T onto the orbit of x obtained by identifying the origin of the real line with x, which is measure-preserving by Proposition 4.13. One can reinterpret the integral $\int_{\mathbb{R}} |f(x+r,x)|\, d\lambda(r)$ as follows:

$$\begin{aligned}\int_{\mathbb{R}} |f(x+r,x)|\, d\lambda(r) &= \lambda\big(\mathbb{R}^{\geq 0} \setminus T_x(\mathbb{R}^{\geq 0})\big) + \lambda\big(T_x(\mathbb{R}^{\geq 0}) \setminus \mathbb{R}^{\geq 0}\big) \\ &= \lambda(\mathbb{R}^{\geq 0} \bigtriangleup T_x(\mathbb{R}^{\geq 0})).\end{aligned}$$

In particular, $T_x \restriction_{\mathbb{R}^{\geq 0}}$ belongs to the commensurating group of $\mathbb{R}^{\geq 0}$. Observe that we also have

$$\int_{\mathbb{R}} f(x+r,x)\, d\lambda(r) = \lambda\big(\mathbb{R}^{\geq 0} \setminus T_x(\mathbb{R}^{\geq 0})\big) - \lambda\big(T_x(\mathbb{R}^{\geq 0}) \setminus \mathbb{R}^{\geq 0}\big),$$

which is equal to the index of $T_x \restriction_{\mathbb{R}^{\geq 0}}$. By Section 6.2, $\mathcal{I}(T_x \restriction_{\mathbb{R}^{\geq 0}}) = \mathcal{I}(T_y \restriction_{\mathbb{R}^{\geq 0}})$ whenever $x\mathcal{R}_{\mathcal{F}}y$. For any $T \in [\mathcal{F}]_1$, we therefore have an $\mathcal{F}$-invariant measurable map $h_T : X \to \mathbb{R}$ given by $h_T(x) = \int_{\mathbb{R}} f(x+r,x)\, d\lambda(r)$. Note that for any $\mathcal{F}$-invariant set $Y \subseteq X$, the mass-transport principle yields

$$\int_Y \rho_T(x)\, d\mu(x) = \int_Y h_T(x)\, d\mu(x). \tag{6.1}$$

Let $(\mathcal{E}, p)$, $X \ni x \mapsto \nu_x \in \mathcal{E}$, be the ergodic decomposition of $(X,\mu,\mathcal{F})$ (see Appendix E.1). Since the map h_T is $\mathcal{F}$-invariant, it produces a map $\tilde{h}_T : \mathcal{E} \to \mathbb{R}$ via $\tilde{h}(\nu) = h(x)$ for any x such that $\nu = \nu_x$ or, equivalently, via

$$\tilde{h}_T(\nu) = \int_X \int_{\mathbb{R}} f(x+r,x)\, d\lambda(r) d\nu(x).$$

Note also that

$$\int_{\mathcal{E}} \big|\tilde{h}_T(\nu)\big|\, dp(\nu) \leq \int_X \int_{\mathbb{R}} |f(x+r,x)|\, d\lambda(r) d\mu(x) = \int_X |\rho_T(x)|\, d\mu(x), \tag{6.2}$$

thus $\tilde{h}_T \in \mathrm{L}^1(\mathcal{E},p,\mathbb{R})$. We can now define the index map of a (possibly non-ergodic) flow as a function $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E},p,\mathbb{R})$.

Definition 6.5. Let $\mathcal{F} = \mathbb{R} \curvearrowright X$ be a free measure-preserving flow on a standard probability space (X,μ); let also $(\mathcal{E},p)$ be the space of $\mathcal{F}$-invariant ergodic probability measures, where p is the probability measure yielding the disintegration of μ. The **index map** is the function $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E},p,\mathbb{R})$ given by

$$\mathcal{I}(T)(\nu) = \tilde{h}_T(\nu) = \int_X \int_{\mathbb{R}} f(x+r,x)\, d\lambda(r) d\nu(x).$$

Proposition 6.6. *For any free measure-preserving flow $\mathcal{F} = \mathbb{R} \curvearrowright X$, the index map $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$ is a continuous and surjective homomorphism. Furthermore, its kernel consists of all $T \in [\mathcal{F}]_1$ satisfying, for almost all $y \in X$,*

$$\lambda_y(\{x \in \operatorname{supp} T : x < y \leq Tx\}) = \lambda_y(\{x \in \operatorname{supp} T : Tx < y \leq x\}). \tag{6.3}$$

Proof. The index map is a homomorphism, since, as we have discussed earlier, $h_T(x)$ is equal to the index of $T_x \restriction_{\mathbb{R}^{\geq 0}}$. Continuity follows from the fact that $\mathcal{I}$ is 1-Lipschitz as a direct consequence of Eq. (6.2). To see surjectivity, pick any $\tilde{h} \in \mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$ and view it as a map $h : X \to \mathbb{R}$ via the identification $h(x) = \tilde{h}(\nu_x)$. Define the automorphism $T \in \mathrm{Aut}(X, \mu)$ by $T(x) = x + h(x)$. It is straightforward to check that $T \in [\mathcal{F}]_1$ and $\mathcal{I}(T) = h$.

Finally, according to the definition of the index map, the kernel is the set of all $T \in [\mathcal{F}]_1$ for which, almost everywhere in X, the condition $h_T(y) = 0$ holds. By the definition of h_T and the charge function f, this translates to the relationship

$$\lambda_y(\{x \in \operatorname{supp} T : x < y < Tx\}) = \lambda_y(\{x \in \operatorname{supp} T : Tx < y < x\}).$$

Since λ_y is atomless, the above equality is equivalent to the desired condition. ∎

The quotient group $[\mathcal{F}]_1/\ker \mathcal{I}$ naturally inherits the quotient norm given by

$$\|T \ker \mathcal{I}\|_1 = \inf_{S \in \ker \mathcal{I}} \|TS\|_1 .$$

By Proposition 6.6, the index map induces an isomorphism between $[\mathcal{F}]_1/\ker \mathcal{I}$ and $\mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$. We argue that this isomorphism is, in fact, an isometry.

Proposition 6.7. *The index map $\mathcal{I}$ induces an isometric isomorphism from $[\mathcal{F}]_1/\ker \mathcal{I}$ onto $\mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$, where the former is endowed with the quotient norm and the latter bears the usual L^1 norm.*

Proof. Since $\int_X |h_T(x)|\, d\mu(x) = \int_{\mathcal{E}} |\tilde{h}_T(\nu)|\, dp(\nu)$, it suffices to show that for all $T \in [\mathcal{F}]_1$

$$\inf_{S \in \ker \mathcal{I}} \|TS\|_1 = \int_X |h_T|\, d\mu.$$

Let $T \in [\mathcal{F}]_1$. We first show the inequality $\inf_{S \in \ker \mathcal{I}} \|TS\|_1 \geq \int_X |h_T|\, d\mu$.

Pick any $S \in \ker \mathcal{I}$. For any $\mathcal{F}$-invariant measurable $Y \subseteq X$, $\int_Y \rho_S\, d\mu = 0$ and

$$\int_Y \rho_{TS}\, d\mu = \int_Y \rho_T(S(x))\, d\mu(x) + \int_Y \rho_S(x)\, d\mu(x) = \int_Y \rho_T\, d\mu = \int_Y h_T\, d\mu,$$

where we rely on Eq. (6.1) and S being measure-preserving. Consider the $\mathcal{F}$-invariant sets

$$Y^{<0} = \{x \in X : h_T(x) < 0\} \quad \text{and} \quad Y^{\geq 0} = \{x \in X : h_T(x) \geq 0\}.$$

The norm $\|TS\|_1$ can be estimated from below as follows.

$$\begin{aligned}\|TS\|_1 &= \int_X |\rho_{TS}|\,d\mu = \int_{Y^{<0}} |\rho_{TS}|\,d\mu + \int_{Y^{\geq 0}} |\rho_{TS}|\,d\mu \\ &\geq \left|\int_{Y^{<0}} \rho_{TS}\,d\mu\right| + \left|\int_{Y^{\geq 0}} \rho_{TS}\,d\mu\right| \\ &= \left|\int_{Y^{<0}} h_T\,d\mu\right| + \left|\int_{Y^{\geq 0}} h_T\,d\mu\right| \\ &= -\int_{Y^{<0}} h_T\,d\mu + \int_{Y^{\geq 0}} h_T\,d\mu = \int_X |h_T|\,d\mu.\end{aligned}$$

We conclude that

$$\inf_{S\in\ker\mathcal{I}} \|TS\|_1 \geq \int_X |h_T|\,d\mu.$$

For the other direction, consider a transformation T' defined by $T'(x) = x + h_T(x)$; note that $T' \in [\mathcal{F}]_1$, $\rho_{T'}(x) = h_{T'}(x) = h_T(x)$ for all $x \in X$, and $T^{-1}T' \in \ker\mathcal{I}$. Therefore

$$\inf_{S\in\ker\mathcal{I}} \|TS\|_1 \leq \left\|TT^{-1}T'\right\|_1 = \|T'\|_1 = \int_X |h_{T'}|\,d\mu = \int_X |h_T|\,d\mu,$$

and the desired equality of norms follows. ∎

Using similar reasoning, we obtain the following characterization of the L^1 full group and the index map, where for all $T \in [\mathcal{R}_\mathcal{F}]$, we let r_T be the measure-preserving transformation of $(\mathcal{R}_\mathcal{F}, M)$ given by $r_T(x,y) = (x, T(y))$ (see Section 4.2).

Proposition 6.8. *Let $\mathcal{F} = \mathbb{R} \curvearrowright X$ be a free measure-preserving $\mathbb{R}$-flow. Consider the set $\mathcal{R}^{\geq 0} = \{(x,y) \in \mathcal{R}_\mathcal{F} : x \geq y\}$. Then for every $T \in [\mathbb{R} \curvearrowright X]$, we have*

$$\|T\|_1 = M\left(\mathcal{R}^{\geq 0} \bigtriangleup r_T(\mathcal{R}^{\geq 0})\right).$$

In particular, the L^1 full group of $\mathcal{F}$ can be seen as the commensurating group of $\mathcal{R}^{\geq 0}$ inside the full group of $\mathcal{R}$. Moreover, in the ergodic case, the index of T as defined above is equal to its index as a commensurating transformation of the set $\mathcal{R}^{\geq 0}$ in the sense of Section 6.1.

Proof. Through the identification $(x,t) \mapsto (x, x+t)$, the measure-preserving transformation r_T is acting on $X \times \mathbb{R}$ as $\mathrm{id}_X \times T_x$, and the set $\mathcal{R}^{\geq 0}$ becomes $X \times \mathbb{R}^{\geq 0}$. We then have

$$\begin{aligned}M(\mathcal{R}^{\geq 0} \bigtriangleup r_T(\mathcal{R}^{\geq 0})) &= \int_X \lambda(\mathbb{R}^{\geq 0} \bigtriangleup (T_x(\mathbb{R}^{\geq 0})))\,d\mu(x) \\ &= \int_X |\rho_T|\,d\mu\end{aligned}$$

by the mass-transport principle, which yields the conclusion, since by the definition of the norm $\|T\|_1 = \int_X |\rho_T| \ d\mu$.

The moreover part follows from a similar computation. ∎

Remark 6.9. The full group of $\mathcal{R}$ embeds via $T \mapsto r_T$ into the group of measure-preserving transformations of $(\mathcal{R}, M)$. One could use this and the fact that the commensurating automorphism group of $\mathcal{R}^{\geq 0}$ is a Polish group in order to give another proof that L^1 full groups of measure-preserving $\mathbb{R}$-flows are themselves Polish.

Chapter 7

Orbitwise ergodic bounded elements of full groups

The purpose of this chapter is to contrast some of the differences in the dynamics of the elements of full groups of $\mathbb{Z}$-actions and those arising from $\mathbb{R}$-flows. Let $S \in [\mathbb{Z} \curvearrowright X]$ be an element of the full group of a measure-preserving aperiodic transformation, and let $\rho_{S^k} : X \to \mathbb{Z}$ be the cocycle associated with S^k for $k \in \mathbb{Z}$. Since $\mathbb{Z}$ is a discrete group, the conservative part in the Hopf decomposition for S (see Appendix C) reduces to the set of periodic orbits. In particular, an aperiodic $S \in [\mathbb{Z} \curvearrowright X]$ has to be dissipative, hence $|\rho_{S^k}(x)| \to \infty$ as $k \to \infty$. When S belongs to the L^1 full group of the action, a theorem of R. M. Belinskaja [8, Thm. 3.2] strengthens this conclusion and asserts that for almost all x in the dissipative component of S, either $\rho_{S^k}(x) \to +\infty$ or $\rho_{S^k}(x) \to -\infty$.

Given an arbitrary free measure-preserving flow $\mathbb{R} \curvearrowright X$, we construct an example of an aperiodic $S \in [\mathbb{R} \curvearrowright X]_1$ for which the signs in $\{\rho_{S^k}(x) : k \in \mathbb{N}\}$ keep alternating indefinitely for almost all $x \in X$. In fact, we present a transformation that acts ergodically on each orbit of the flow (in particular, it is conservative and globally ergodic as soon as the flow is ergodic). Moreover, we ensure it has a uniformly bounded cocycle. Our argument uses a variant of the well-known cutting and stacking construction adapted for infinite measure spaces. Additional technical difficulties arise from the necessity to work across all orbits of the flow simultaneously. The transformation will arise as a limit of special partial transformations we call *castles*, which we now define.

The **pseudo full group** of the flow is the set of injective Borel maps $\varphi : \operatorname{dom}\varphi \to \operatorname{rng}\varphi$ between Borel sets $\operatorname{dom}\varphi \subseteq X$, $\operatorname{rng}\varphi \subseteq X$, for which there exists a countable Borel partition $(A_n)_{n\in\mathbb{N}}$ of the domain $\operatorname{dom}\varphi$ and a countable family of reals $(t_n)_{n\in\mathbb{N}}$ such that $\varphi(x) = x + t_n$ for every $x \in A_n$. Such maps are measure-preserving isomorphisms between $(\operatorname{dom}\varphi, \mu\restriction_{\operatorname{dom}\varphi})$ and $(\operatorname{rng}\varphi, \mu\restriction_{\operatorname{rng}\varphi})$, in other words they are partial transformations. The **support** of φ is the set

$$\operatorname{supp}\varphi = \{x \in \operatorname{dom}\varphi : \varphi(x) \neq x\} \cup \{x \in \operatorname{rng}\varphi : \varphi^{-1}(x) \neq x\}.$$

Given φ in the pseudo full group and a Borel set $A \subseteq X$, we let

$$\varphi(A) = \{\varphi(x) : x \in A \cap \operatorname{dom}\varphi\}.$$

In particular, $\varphi(A) = \varnothing$ if A is disjoint from $\operatorname{dom}\varphi$. A **castle** is an element φ of the pseudo full group of the flow such that for $B = \operatorname{dom}\varphi \setminus \operatorname{rng}\varphi$ the sequence $(\varphi^k(B))_{k\in\mathbb{N}}$ consists of pairwise disjoint subsets which cover its support. Since φ is measure-preserving, for almost every $x \in B$ there is $k \in \mathbb{N}$ such that $\varphi^k(x) \notin \operatorname{dom}\varphi$. It follows that φ^{-1} is also a castle. The set B is called the **basis** of the castle, and the basis of its inverse C is called its **ceiling**, which is equal to $\operatorname{rng}\varphi \setminus \operatorname{dom}\varphi$. Observe that if two castles have disjoint supports, then their union is also a castle. We denote by $\vec{\varphi} : B \to C$ the element of the pseudo full group which takes every element of the basis of φ to the corresponding element of the ceiling.

Remark 7.1. Equivalently, one could define a castle as an element φ of the pseudo full group which induces a *graphing* consisting of finite segments only (see [33, Sec. 17] for the definition of a graphing). It induces a partial order $\leq_\varphi$ defined by $x \leq_\varphi y$ if and only if there is $k \in \mathbb{N}$ such that $y = \varphi^k(x)$. The basis of the castle is the set of minimal elements, while the ceiling is the set of maximal ones. Finally, $\vec{\varphi}$ is the map which takes a minimal element to the unique maximal element above it.

Theorem 7.2. *Let $\mathbb{R} \curvearrowright X$ be a free measure-preserving flow. There exists $S \in [\mathbb{R} \curvearrowright X]$ that acts ergodically on every orbit of the flow and whose cocycle is bounded by* 4. *Moreover, the signs in $\{\rho_{S^k}(x) : k \in \mathbb{N}\}$ keep changing indefinitely for almost all $x \in X$.*

Proof. Fix a free measure-preserving flow $\mathbb{R} \curvearrowright X$, and let $C \subset X$ be a cross-section.

We recall some notation from Section 1.2.4. Since C is lacunary, for any $c \in C$, the function $\mathrm{gap}_C(c) = \min\{r > 0 : c + r \in C\}$ is well-defined. This gives the first return map $\sigma_C : C \to C$ via $\sigma_C(c) = c + \mathrm{gap}_C(c)$, which is Borel. There is also a natural bijective correspondence between X and the set $\{(c,t) \in C \times \mathbb{R}^{\geq 0} : c \in C, 0 \leq t < \mathrm{gap}_C(c)\}$. Let λ_c^C be the "Lebesgue measure" on $c + [0, \mathrm{gap}_C(c))$ given by

$$\lambda_c^C(A) = \lambda(\{r \in \mathbb{R} : 0 \leq r < \mathrm{gap}_C(c), c + r \in A\}).$$

The measure μ on X can then be disintegrated as $\mu(A) = \int_C \lambda_c^C(A)\, d\nu(c)$ for some finite (but not necessarily probability) measure ν on C, as explained at the end of Section 1.2.4.

Let $(C_n)_{n\in\mathbb{N}}$ be a vanishing sequence of markers—a sequence of nested cross-sections $C_1 \supset C_2 \supset C_3 \cdots$ with an empty intersection: $\bigcap_{n\in\mathbb{N}} C_n = \varnothing$. We may arrange C_1 to be such that $\mathrm{gap}_{C_1}(c) \in (2,3)$ for all $c \in C_1$. Put

$$C_0 = \{c + k : c \in C_1, k \in \{0,1,2\}\}$$

and $Y = C_1 + [0,2)$. Note that $\mu(X \setminus Y) \leq \frac{1}{3}$. Our first goal is to define an element φ of the pseudo full group with domain and range equal to Y such that for almost every $x \in Y$, the action of φ on the intersection of the orbit of x with Y is ergodic and has a cocycle bounded by 3. It will then be easy to modify φ to an element of the full group whose action on each orbit of the flow is ergodic at the cost of increasing the cocycle bound to 4.

Our first partial transformation φ will arise as the limit of a sequence of castles $(\varphi_n)_{n\in\mathbb{N}}$, with each φ_n belonging to the pseudo full group of $\mathcal{R}_{C_n}$. We also use another family of castles $(\psi_n)_{n\in\mathbb{N}}$ which allows us to extend φ_n by "going back" from its ceiling to its basis while keeping the cocycle bound (this is our main adjustment compared to the usual cutting and stacking procedure). Both sequences of castles will have their cocycles bounded by 3. Here are the basic constraints that these sequences have to satisfy:

(1) for all $n \geq 1$, $Y = \operatorname{supp} \varphi_n \sqcup \operatorname{supp} \psi_n$;

(2) for all $n \geq 1$, φ_{n+1} extends φ_n;

(3) $\mu(\operatorname{supp} \psi_n)$ tends to 0 as n tends to $+\infty$.

The bases and ceilings of $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ will satisfy additional constraints that will enable us to make the induction work and ensure ergodicity on each orbit of the flow. In order to specify these constraints properly, we introduce the following notation.

Each orbit of the flow comes with the linear order $<$ inherited from $\mathbb{R}$ via $x < y$ if and only if $y = x + t$ for some $t > 0$. Set $\kappa_{C_n}(x)$ to be the minimum of the intersection of C_n with the cone $\{y \in X : y \geq x\}$.

Let $\mathcal{D}_1 = C_1 + 2 \subseteq C_0$ and $\mathcal{D}_n$ be the set of those $x \in \mathcal{D}_1$ which are maximal in $\kappa_{C_n}^{-1}(c)$ among points of $\mathcal{D}_1$ for some $c \in C_n$; in other words,

$$\mathcal{D}_n = \{x \in \mathcal{D}_1 : (x, \kappa_{C_n}(x)) \cap C_0 = \varnothing\}.$$

Note that by construction, the distance between x and $\kappa_{C_n}(x)$ is less than 1 for each $x \in \mathcal{D}_n$. Let ι_n be the map $C_n \to \mathcal{D}_n$ which assigns to $c \in C_n$ the $<$-least element of $\mathcal{D}_n$ that is greater than c.

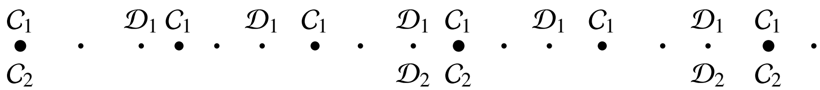

Figure 7.1. An example of cross-sections C_0 (all points), C_1 (dots of size • and above), C_2 (marked as ●) and $\mathcal{D}_1$, $\mathcal{D}_2$.

The bases and ceilings of φ_n and ψ_n are as follows:

- the basis of φ_n is $A_n = C_n + \left[0, \frac{1}{2^n}\right)$;
- the ceiling of φ_n is $B_n = \mathcal{D}_n + \left[-\frac{1}{2} - \frac{1}{2^n}, -\frac{1}{2}\right)$;
- the basis of ψ_n is $E_n = \mathcal{D}_n + \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^n}\right)$;
- the ceiling of ψ_n is $F_n = C_n + \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}\right)$.

Furthermore, we impose two **translation conditions**, which help us to preserve the above concrete definitions of the bases and ceilings at the inductive step when we construct φ_{n+1} and ψ_{n+1}:

- $\vec{\varphi}_n(c+t) = \iota_n(c) + t - \frac{1}{2} - \frac{1}{2^n}$ for all $c \in C_n$ and all $t \in \left[0, \frac{1}{2^n}\right)$.
- $\vec{\psi}_n(d+t) = \iota_n^{-1}(d) + t + 1$ for all $d \in \mathcal{D}_n$ and all $t \in \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^n}\right)$.

The first step of the construction consists of the castle $\varphi_1 : x \mapsto x+1$, which has basis $A_1 = C_1 + [0, \frac{1}{2})$ and ceiling $B_1 = \mathcal{D}_1 + [-1, -\frac{1}{2})$, and the castle $\psi_1 : x \mapsto x - 1$ with basis $E_1 = \mathcal{D}_1 + [-\frac{1}{2}, 0)$ and ceiling $F_1 = C_1 + [\frac{1}{2}, 1)$.

We now concentrate on the induction step: suppose φ_n and ψ_n have been built for some $n \geq 1$; let us construct φ_{n+1} and ψ_{n+1}.

The strategy is to split the bases of φ_n and ψ_n into two equal intervals and "interleave" the "two halves" of φ_n with "one half" of ψ_n followed by "gluing" adjacent ceilings and bases within the same C_{n+1} segment (see Figure 7.2). To this end, we introduce two intermediate castles $\tilde{\varphi}_n$ and $\tilde{\psi}_n$ that will ensure that φ_{n+1} "wiggles" more than φ_n, yielding ergodicity of the final transformation.

Define two new half-measure subsets of the bases A_n and E_n respectively:

- $A_n^1 = C_n + \left[0, \frac{1}{2^{n+1}}\right)$;
- $E_n^0 = \mathcal{D}_n + \left[-\frac{1}{2} + \frac{1}{2^{n+1}}, -\frac{1}{2} + \frac{1}{2^n}\right)$;

and let

$$B_n^0 = \vec{\varphi}_n(A_n^1) = \mathcal{D}_n + \left[-\frac{1}{2} - \frac{1}{2^n}, -\frac{1}{2} - \frac{1}{2^{n+1}}\right),$$

and

$$F_n^0 = \vec{\psi}_n(E_n^0) = C_n + \left[\frac{1}{2} + \frac{1}{2^{n+1}}, \frac{1}{2} + \frac{1}{2^n}\right),$$

where the two equalities are consequences of the translation conditions. Let G_n be the ψ_n-saturation of E_n^0, and note that the restriction of ψ_n to G_n is a castle with support G_n, whose basis is E_n^0 and whose ceiling is F_n^0. Finally, let

$$A_n^0 = A_n \setminus A_n^1 = C_n + \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right).$$

We define the partial transformation $\xi_n : B_n^0 \sqcup F_n^0 \to E_n^0 \sqcup A_n^0$ to be used for "gluing together" φ_n and the restriction of ψ_n to G_n:

- $\xi_n(b) = b + \frac{3}{2^{n+1}} \in E_n^0$ for all $b \in B_n^0$ and
- $\xi_n(f) = f - \frac{1}{2} \in A_n^0$ for all $f \in F_n^0$.

Set $\tilde{\varphi}_n = \varphi_n \sqcup \xi_n \sqcup \psi_{n\restriction G_n}$, whereas $\tilde{\psi}_n$ is simply the restriction of ψ_n onto the complement of G_n. Observe that $\tilde{\varphi}_n$ has basis A_n^1 and ceiling

$$B_n^1 = B_n \setminus B_n^0 = \mathcal{D}_n + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right),$$

while $\tilde{\psi}_n$ has basis

$$E_n^1 = E_n \setminus E_n^0 = \mathcal{D}_n + \left[-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2^{n+1}}\right)$$

and ceiling

$$F_n^1 = F_n \setminus F_n^0 = C_n + \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}}\right).$$

We continue to have $Y = \operatorname{supp} \tilde{\varphi}_n \sqcup \operatorname{supp} \tilde{\psi}_n$, but the support of $\tilde{\psi}_n$ is half the support of ψ_n, meaning that $\mu(\operatorname{supp} \tilde{\psi}_n) = \frac{1}{2}\mu(\operatorname{supp} \psi_n)$.

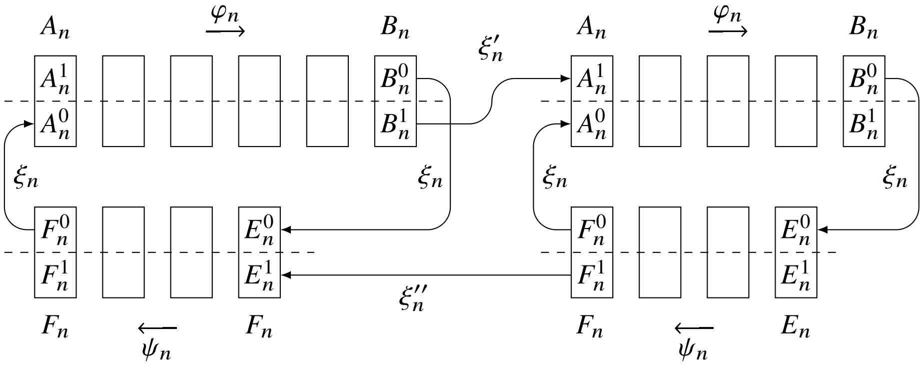

Figure 7.2. Inductive step.

The ceiling of $\tilde{\varphi}_n$ is equal to $B_n^1 = \mathcal{D}_n + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$, whereas we need the ceiling of φ_{n+1} to be equal to $B_{n+1} = \mathcal{D}_{n+1} + \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$. We obtain the required φ_{n+1} and ψ_{n+1} out of $\tilde{\varphi}_n$ and $\tilde{\psi}_n$, respectively, by "passing through each element of $C_n \setminus C_{n+1}$ ".

Note that $\mathcal{D}_{n+1}$ is equal to the set of $d \in \mathcal{D}_n$ such that $\kappa_{C_n}(d) \in C_{n+1}$. Each $x \in B_n^1 \setminus B_{n+1}$ can be written uniquely as $x = d + t$ where $d \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$ and $t \in \left[-\frac{1}{2} - \frac{1}{2^{n+1}}, -\frac{1}{2}\right)$. Set

$$\xi_n'(x) = \kappa_{C_n}(d) + t + \frac{1}{2} + \frac{1}{2^{n+1}},$$

and note that $\xi_n'(x)$ belongs to $(C_n \setminus C_{n+1}) + \left[0, \frac{1}{2^{n+1}}\right) = A_n^1 \setminus A_{n+1}$, hence ξ_n' is a measure-preserving bijection from $B_n^1 \setminus B_{n+1}$ onto $A_n^1 \setminus A_{n+1}$.

The transformation φ_{n+1} is set to be $\tilde{\varphi}_n \sqcup \xi_n'$, and we claim that it is a castle with basis A_{n+1} and ceiling B_{n+1}. This amounts to showing that for all $x \in A_{n+1}$, there is $k \in \mathbb{N}$ such that $\varphi_{n+1}^k(x)$ is not defined. Pick $x \in A_{n+1}$ and write it as $c_0 + t$ for some

$c_0 \in C_{n+1}$ and $t \in [0, \frac{1}{2^{n+1}})$. Let c_1 be the successor of c_0 in C_n, which we suppose is not an element of C_{n+1}. By the construction of $\tilde{\varphi}_n$ and ξ'_n, there is $k \in \mathbb{N}$ such that $\xi'_n(\tilde{\varphi}_n^k(x)) \in c' + [0, \frac{1}{2^{n+1}})$, which means that $\varphi_{n+1}^{k+1}(x) \in c' + [0, \frac{1}{2^{n+1}})$. Iterating this argument, we eventually find $k_0, p \in \mathbb{N}$ such that $\varphi_{n+1}^{k_0}(x) \in c_p + [0, \frac{1}{2^{n+1}})$ for some $c_p \in C_n$ such that the successor c_{p+1} of c_p in C_n belongs to C_{n+1}. By the definition of $\tilde{\varphi}_n$, we must have some $l \in \mathbb{N}$ such that $\varphi_{n+1}^{k_0+l}(x) = \tilde{\varphi}_n^l(\varphi_{n+1}^{k_0}(x)) \in B_{n+1}$, whereas $\varphi_{n+1}^{k_0+l+1}(x)$ is not defined. Thus, φ_{n+1} is indeed a castle.

The extension ψ_{n+1} of $\tilde{\psi}_n$ is defined similarly by connecting adjacent segments of F_n^1 and E_n^1 by a translation. More specifically, each $x \in F_n^1 \setminus F_{n+1}$ can be written uniquely as $x = c + t$ for some $c \in C_n \setminus C_{n+1}$ and $t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}})$. The restriction of κ_{C_n} to $\mathcal{D}_n$ is a bijection $\mathcal{D}_n \to C_n$. We denote its inverse by p_n and let $\xi''_n(x) = p_n(c) + t - 1$. The map $\psi_{n+1} = \tilde{\psi}_n \sqcup \xi''_n$ can be checked to be a castle with basis E_{n+1} and ceiling F_{n+1} as desired. It also follows that the translation conditions continue to be satisfied by both φ_{n+1} and ψ_{n+1}.

The transformations φ_n extend each other, so $\varphi = \bigcup_n \varphi_n$ is an element of the pseudo full group supported on $Y = \operatorname{supp} \varphi_n \sqcup \operatorname{supp} \psi_n$. Note also that

$$\mu(\operatorname{supp} \psi_{n+1}) = \mu(\operatorname{supp} \psi_n)/2,$$

and therefore $\operatorname{dom} \varphi = Y = \operatorname{rng} \varphi$. We claim that φ, seen as a measure-preserving transformation of Y, induces an ergodic measure-preserving transformation on $(y + \mathbb{R}) \cap Y$ for almost all $y \in Y$, where $y + \mathbb{R}$ is endowed with the Lebesgue measure. This follows from the fact that φ induces a **rank-one** transformation of the infinite measure space $(y + \mathbb{R}) \cap Y$: for all Borel $A \subseteq (y + \mathbb{R}) \cap Y$ of finite Lebesgue measure and all $\epsilon > 0$, there are $B \subseteq (y + \mathbb{R}) \cap Y$, $k \in \mathbb{N}$, and a subset $F \subseteq \{0, \dots, k\}$ such that $B, \varphi(B), \dots, \varphi^k(B)$ are pairwise disjoint and

$$\lambda(A \triangle (\bigsqcup_{f \in F} \varphi^f(B))) < \epsilon.$$

Indeed, at each step n for every $c \in C_n$, the iterates of $c + [0, \frac{1}{2^n})$ by the restriction of φ_n to the interval $[c, \iota_n(c))$ are disjoint "intervals of size 2^{-n}", i.e., sets of the form $t + [0, \frac{1}{2^n})$, and these iterates cover a proportion $1 - \frac{1}{2^n}$ of $[c, \iota_n(c))$ (the rest of this interval being $[c, \iota_n(c)) \cap \operatorname{supp} \psi_n$).

It remains to extend φ supported on Y to a measure-preserving transformation S with $\operatorname{supp} S = X$. Let $Z = X \setminus Y$ be the leftover set,

$$Z = \{c + t : c \in C_1 : 2 \leq t < \operatorname{gap}_{C_1}(c)\},$$

and put

$$Z' = \{c + t : c \in C_1,\ 2 - \operatorname{gap}_{C_1}(c) \leq t < 2\}.$$

Figure 7.3 illustrates an interval between $c \in C_1$ and $c' = \sigma_{C_1}(c)$. Within this gap, Z corresponds to $[c + 2, c + 2 + \operatorname{gap}_{C_1}(c))$, and Z' is an interval of the exact same length adjacent to it on the left. Note that $Z' \subseteq Y$ by construction. Let $\eta : Z' \to Z$ be the natural translation map, $\eta(x) = x + \mathrm{gap}_{C_1}(c)$ for all $x \in Z'$ satisfying $x \in c + [0, \mathrm{gap}_{C_1}(c))$. Observe that η is a measure-preserving bijection, and its cocycle is bounded by 1.

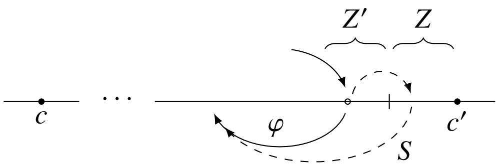

Figure 7.3. Construction of the transformation S.

We now rewire the orbits of φ and define $S : X \to X$ as follows (see Figure 7.3):

$$S(x) = \begin{cases} \varphi(x) & \text{if } x \notin Z \cup Z'; \\ \eta(x) & \text{if } x \in Z'; \\ \varphi(\eta^{-1}(x)) & \text{if } x \in Z. \end{cases}$$

It is straightforward to verify that S is a free measure-preserving transformation, and the distance $D(x, Sx) \leq 4$ for all $x \in X$ because $|\rho_\varphi(x)| \leq 3$ and $|\rho_\eta(x)| \leq 1$ for all x in their domains. Note that for every $y \in Y$, the intersection of the S-orbit with Y coincides with its φ-orbit. Since φ is ergodic on each orbit of the flow intersected with Y, and considering that $X = Y \sqcup Z$ and $S^{-1}(Z) \subseteq Y$, it follows that S is ergodic on every orbit of the flow. Therefore, S satisfies the conclusion of the theorem. ■

Remark 7.3. The bound 4 in the formulation of Theorem 7.2 is of no significance, as by rescaling the flow, it can be replaced with any $\epsilon > 0$.

Chapter 8

Conservative and intermitted transformations

Interesting dynamics of conservative transformations is present only in the non-discrete case, as it reduces to periodicity for countable group actions. Chapter 7 provides an illustrative construction of a conservative automorphism and shows that they exist in L^1 full groups of all free flows. The present chapter is devoted to the study of such elements. The central role is played by the concept of an intermitted transformation, which is related to the notion of induced transformation. Using this tool, we show that all conservative elements of $[\mathbb{R} \curvearrowright X]_1$ can be approximated by periodic automorphisms, and hence belong to the derived L^1 full group of $\mathbb{R} \curvearrowright X$; see Corollary 8.8.

Throughout the chapter, we fix a free measure-preserving flow $\mathbb{R} \curvearrowright X$ on a standard Lebesgue space (X, μ). Given a cross-section $\mathcal{C} \subset X$, recall that we defined an equivalence relation $\mathcal{R}_\mathcal{C}$ by declaring $x\mathcal{R}_\mathcal{C}y$ whenever there is $c \in \mathcal{C}$ such that both x and y belong to the gap between c and $\sigma_\mathcal{C}(c)$. More formally, $x\mathcal{R}_\mathcal{C}y$ if there is $c \in \mathcal{C}$ such that $\rho(c, x) \geq 0$, $\rho(c, y) \geq 0$ and $\rho(x, \sigma_\mathcal{C}(c)) > 0$, $\rho(y, \sigma_\mathcal{C}(c)) > 0$. Such an equivalence relation is smooth.

Now let $T \in [\mathbb{R} \curvearrowright X]$ be a conservative transformation. Under the action of T, almost every point returns to its $\mathcal{R}_\mathcal{C}$-class infinitely often, which suggests the idea of the first return map.

Definition 8.1. The **intermitted transformation** $T_{\mathcal{R}_\mathcal{C}} : X \to X$ is defined by

$$T_{\mathcal{R}_\mathcal{C}}x = T^{n(x)}x, \quad \text{where } n(x) = \min\{n \geq 1 : x\mathcal{R}_\mathcal{C}T^{n(x)}x\}.$$

The map $T_{\mathcal{R}_\mathcal{C}}$ is well-defined, since T is conservative, and it preserves the measure μ, since $T_{\mathcal{R}_\mathcal{C}}$ belongs to the full group of T.

Remark 8.2. The concept of an intermitted transformation T_E makes sense for any equivalence relation E for which the intersection of any orbit of T with any E-class is either empty or infinite. In particular, intermitted transformations can be considered for any conservative $T \in [G \curvearrowright X]$ in a full group of a locally compact group action. For instance, with a cocompact cross-section $\mathcal{C}$ we can associate an equivalence relation of lying in the same cell of the Voronoi tessellation (see Appendix E.2). Such an equivalence relation does have the aforementioned transversal property, and hence the intermitted transformation is well-defined.

Note also the following connection with the more familiar construction of the induced transformation. Let $T \in \mathrm{Aut}(X, \mu)$, and let $A \subseteq X$ be a set of positive measure. Recall that the induced transformation $T_A \in \mathrm{Aut}(X, \mu)$ is supported on the set A and is defined for $x \in A$ by $T_Ax = T^{n(x)}x$, where $n(x) = \min\{n \geq 1 : T^nx \in A\}$. Define $\mathcal{A}$ to be the equivalence relation with two classes: A and $X \setminus A$. The induced transformations T_A and $T_{X \setminus A}$ commute and satisfy $T_A \circ T_{X\setminus A} = T_{\mathcal{A}}$.

The next lemma forms the core of this chapter. It shows that the operation of taking an intermitted transformation does not increase the norm. As we discuss later in Remark 8.5, the analog of this statement is false even for $\mathbb{R}^2$-flows, which perhaps justifies the technical nature of the argument.

Lemma 8.3. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative automorphism, and let C be a cross-section. Let also Y be the set of points where T and $T_{\mathcal{R}_C}$ differ: $Y = \{x \in X : Tx \neq T_{\mathcal{R}_C} x\}$. One has $\int_Y |\rho_{T_{\mathcal{R}_C}}|\, d\mu \leq \int_Y |\rho_T|\, d\mu$.*

Proof. By the definition of Y, for any $x \in Y$, the arc from x to Tx crosses at least one point of C. We may therefore represent $|\rho_T(x)|$ as the sum of the distance from x to the first point of C along the arc plus the rest of the arc. More formally, for $x \in X$, let $\pi_C(x)$ be the unique $c \in C$ such that $x \in c + [0, \mathrm{gap}_C(c))$. Define $\alpha : Y \to \mathbb{R}^{\geq 0}$ by

$$\alpha(x) = \begin{cases} |\rho(x, \sigma_C(\pi_C(x)))|, & \text{if } \rho(x, Tx) > 0, \\ |\rho(x, \pi_C(x))| & \text{if } \rho(x, Tx) < 0. \end{cases}$$

Note that $\alpha(x) \leq |\rho_T(x)|$, and set $\beta(x) = |\rho_T(x)| - \alpha(x)$, so that

$$\int_Y |\rho_T|\, d\mu = \int_Y \alpha\, d\mu + \int_Y \beta\, d\mu.$$

For instance, in the context of Figure 8.1, $\alpha(x_4) = \rho(x_4, c_2)$ and $\beta(x_4) = \rho(c_2, x_5)$. Let us partition $Y = Y' \sqcup Y''$, where

$$Y' = \big\{x \in Y : \rho(x, Tx) \text{ and } \rho(x, T_{\mathcal{R}_C} x) \text{ have the same sign or } T_{\mathcal{R}_C} x = x\big\},$$

and $Y'' = Y \setminus Y'$ consists of those $x \in Y$ for which the signs of $\rho(x, Tx)$ and $\rho(x, T_{\mathcal{R}_C} x)$ are different. For example, referring to the same figure, $x_0 \in Y''$, while $x_2 \in Y'$.

To prove the lemma, it is enough to show two inequalities:

$$\int_{Y'} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) \leq \int_Y \alpha(x)\, d\mu(x), \tag{8.1}$$

$$\int_{Y''} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) \leq \int_Y \beta(x)\, d\mu(x). \tag{8.2}$$

Eq. (8.1) is straightforward, since the equality of signs of $\rho(x, Tx)$ and $\rho(x, T_{\mathcal{R}_C} x)$ implies that $T_{\mathcal{R}_C} x$ is closer than x to the point $c \in C$, which is crossed by the arc from x to Tx. For example, the point x_2 in Figure 8.1 satisfies

$$|\rho_{T_{\mathcal{R}_C}}(x_2)| = \rho(x_2, x_4) \leq \rho(x_2, c_2) = \alpha(x_2).$$

Thus $|\rho_{T_{\mathcal{R}_C}}(x)| \le \alpha(x)$ for all $x \in Y'$, and so

$$\int_{Y'} |\rho_{T_{\mathcal{R}_C}}|\, d\mu \le \int_{Y'} \alpha\, d\mu \le \int_{Y} \alpha\, d\mu,$$

which establishes (8.1). The other inequality will require a bit more work.

For $x \in Y''$, let $N(x) \ge 1$ be the smallest integer such that the sign of $\rho(x, T^{N(x)+1}x)$ is opposite to that of $\rho_T(x)$. In less formal terms, $N(x)$ is the smallest integer such that the arc from $T^{N(x)}x$ to $T^{N(x)+1}x$ jumps over x. In particular, points $T^k x$, $1 \le k \le N(x)$, are all on the same side relative to x, while $T^{N(x)+1}x$ is on the other side of it. We consider the map $\eta : Y'' \to X$ given by $\eta(x) = T^{N(x)}x$. The properties of this map will be crucial for establishing the inequality (8.2), so let us provide some explanations first.

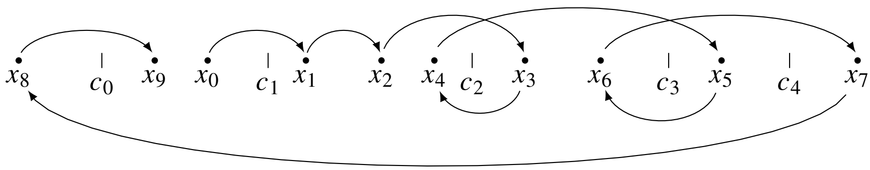

Figure 8.1. Dynamics of a conservative orbit.

Consider once again Figure 8.1, which shows a partial orbit of a point x_0 for $x_i = T^i x_0$ up to $i \le 9$ and several points $c_i \in C$. First, as we have already noted before, $x_0 \in Y$, since $\neg x_0 \mathcal{R}_C x_1$; moreover, $x_0 \in Y''$, since $x_9 = T_{\mathcal{R}_C} x_0$ is to the left of x_0, while x_1 is to the right of it, so $\rho(x_0, x_1)$ and $\rho(x_0, x_9)$ have opposite signs. Also, $N(x_0) = 7$, because x_8 is the first point in the orbit to the left of x_0, thus $\eta(x_0) = x_7$. In general, we have $T^{N(x)+1}x \ne T_{\mathcal{R}_C}x$. However, the equality $T^{N(x)+1}x = T_{\mathcal{R}_C}x$ holds when $x \in Y''$ and the points $T^{N(x)+1}x$ and x are $\mathcal{R}_C$-equivalent.

The next point in the orbit x_1 is not in Y, whereas $x_2 \in Y$ but $x_2 \notin Y''$, because $T_{\mathcal{R}_C} x_2 = x_4$ and both $\rho(x_2, x_3)$ and $\rho(x_2, x_4)$ are positive. The point x_3 belongs to Y'' and has $N(x_3) = 1$ with $\eta(x_3) = x_4$. The points x_4, x_5, x_6 are in Y, but whether any of them are elements of Y'' is not clear from Figure 8.1, as the orbit segment is too short to clarify the values of $T_{\mathcal{R}_C} x_i$, $i = 4, 5, 6$. However, if x_4, x_5, x_6 happen to lie in Y'', then $N(x_5) = 1$ with $\eta(x_5) = x_6$, and $N(x_4) = 3$, $N(x_6) = 1$, $\eta(x_4) = \eta(x_6) = x_7 = \eta(x_0)$. In particular, the function $x \mapsto \eta(x)$ is not necessarily one-to-one, but we are going to argue that it is always finite-to-one.

Claim 1. *If $x, y \in Y''$ are distinct points such that $\eta(x) = \eta(y)$, then $\neg x \mathcal{R}_C y$.*

Proof of the claim. Suppose $x, y \in Y''$ satisfy $\eta(x) = \eta(y)$. The definition of η implies that x and y must belong to the same orbit of T, and we may assume without loss of generality that $y = T^{k_0}x$ for some $k_0 \ge 1$. If the orbit of x and y is aperiodic, it implies that $N(x) > k_0$ and $N(y) + k_0 = N(x)$, $N(y) \ge 1$. However, even if the orbit is periodic, either $N(y) + k_0 = N(x)$ for the smallest positive integer k_0 such that $y = T^{k_0}x$ or $N(x) + k_0' = N(y)$ for the smallest positive integer k_0' such that $x = T^{k_0'}y$. Interchanging the roles of x and y if necessary, we may therefore assume that $N(y) + k_0 = N(x)$ holds for some $k_0 \geq 1$, $T^{k_0}x = y$, regardless of the type of orbit we consider.

Suppose x and y are $\mathcal{R}_C$-equivalent. Let $k \geq 1$ be the smallest natural number for which x and $T^k x$ are $\mathcal{R}_C$-equivalent. By the assumption $x\mathcal{R}_C y$ and the choice of k_0 we have $k \leq k_0 < N(x)$. By the definition of $N(x)$, all points $T^i x$, $1 \leq i \leq N(x)$, are on the same side of x. In particular, this applies to Tx and $T^k x$, which shows that $\rho(x, Tx)$ and $\rho(x, T_{\mathcal{R}_C} x)$ have the same sign, thus $x \notin Y''$. $\square_{\text{claim}}$

The above claim implies that the function $x \mapsto \eta(x)$ is finite-to-one, for the arc from $\eta(x)$ to $T\eta(x)$ intersects only finitely many $\mathcal{R}_C$-equivalence classes, and the preimage of $\eta(x)$ picks at most one point from each such class. Note also that $\eta(x) \in Y$ for all $x \in Y''$, but $\eta(x)$ may not be an element of Y''. Among the $\mathcal{R}_C$-equivalence classes that the arc from $\eta(x)$ to $T\eta(x)$ crosses, two are special: the intervals that contain $T\eta(x)$ and $\eta(x)$, respectively. Our goal will be to bound the sum of $|\rho_{T_{\mathcal{R}_C}}(x)|$ over the points x with the same $\eta(x)$ value by $\beta(\eta(x))$ (see Claim 3 below). For a typical point x, we can bound $|\rho_{T_{\mathcal{R}_C}}(x)|$ simply by the length of the interval of its $\mathcal{R}_C$-class. For example, Figure 8.1 does not specify $T_{\mathcal{R}_C} x_4$, but we can be sure that $|\rho_{T_{\mathcal{R}_C}}(x_4)| \leq \rho(c_1, c_2)$. In view of Claim 1, such an estimate comes close to showing that the sum of $|\rho_{T_{\mathcal{R}_C}}(x)|$ over x with the same image $\eta(x)$ is bounded by $|\rho(\eta(x), T\eta(x))|$. It merely comes close due to the two special $\mathcal{R}_C$-classes mentioned above, where our estimate needs to be improved. The next claim shows that one of these special cases is of no concern as x is never $\mathcal{R}_C$-equivalent to $\eta(x)$.

Claim 2. *For all $x \in Y''$, we have $\neg x\mathcal{R}_C \eta(x)$.*

Proof of the claim. Suppose, towards a contradiction, that $x\mathcal{R}_C\eta(x)$, and let $k \geq 1$ be the smallest integer for which $x\mathcal{R}_C T^k(x)$; in particular, $T_{\mathcal{R}_C} x = T^k x$. Note that $k \leq N(x)$ by the assumption, and by the definition of $N(x)$, $\rho(T^k x, x)$ has the same sign as $\rho_T(x)$, whence $x \notin Y''$. $\square_{\text{claim}}$

Pick some $y \in Y$ with non-empty preimage $\eta^{-1}(y)$, and let $z_1, \ldots, z_n \in Y''$ be all the elements in $\eta^{-1}(y)$. For instance, in the situation depicted in Figure 8.1, we may have $n = 3$ and $z_1 = x_0$, $z_2 = x_4$, $z_3 = x_6$, and $y = x_7$. The following claim unlocks the path toward the inequality (8.2).

Claim 3. *In the above notation, $\sum_{i=1}^n |\rho_{T_{\mathcal{R}_C}}(z_i)| \leq \beta(y)$.*

Proof of the claim. Recall that the arc from y to Ty crosses at least one point in C. If $c \in C$ is the closest to y among such points, then $\beta(y)$ is defined to be $|\rho(c, Ty)|$. For instance, in the notation of Figure 8.1, $\beta(x_7) = |\rho(c_4, x_8)|$. Each point z_i is located under the arc from y to Ty, and by Claim 2, no point z_i belongs to the interval from c to y. In the language of our concrete example, no point z_i can be between c_4 and x_7.

As discussed before, $|\rho_{T_{\mathcal{R}_C}}(x)|$ is always bounded by the length of the gap to which x belongs. This is sufficient to prove the claim if no z_i is equivalent to Ty, as in this case the whole $\mathcal{R}_C$-equivalence class of every z_i is fully contained under the interval between c and Ty, and distinct z_i represent distinct $\mathcal{R}_C$-classes by Claim 1. This is the situation depicted in Figure 8.1, and our argument boils down to the inequalities

$$\begin{aligned}|\rho_{T_{\mathcal{R}_C}}(x_0)| + |\rho_{T_{\mathcal{R}_C}}(x_4)| + |\rho_{T_{\mathcal{R}_C}}(x_5)| &\leq |\rho(c_0, c_1)| + |\rho(c_1, c_2)| + |\rho(c_2, c_3)| \\ &\leq |\rho(c_0, c_4)| \leq \beta(x_7).\end{aligned}$$

Suppose there is some z_i such that $z_i \mathcal{R}_C Ty$. By Claim 1, such z_i must be unique, and we assume without loss of generality that $z_1 \mathcal{R}_C Ty$. For example, this situation would occur if in Figure 8.1 Tx_7 were equal to x_9. Let c' be the first element of C over which goes the arc from z_1 to Tz_1 (it would be the point c_1 in Figure 8.1). It is enough to show that $|\rho_{T_{\mathcal{R}_C}}(z_1)| \leq |\rho(T_{\mathcal{R}_C} z_1, c')|$, as we can use the previous estimate for all other $|\rho_{T_{\mathcal{R}_C}}(z_i)|$, $i \geq 2$. Note that $T_{\mathcal{R}_C} z_1 = Ty$, and $z_1 \in Y''$ by assumption, which implies that the signs of $\rho(z_1, T_{\mathcal{R}_C} z_1)$ and $\rho(z_1, c')$ are different. The latter is equivalent to saying that z_1 is between $T_{\mathcal{R}_C} z_1$ and c', i.e., $|\rho(T_{\mathcal{R}_C} z_1, c')| = |\rho_{T_{\mathcal{R}_C}}(z_1)| + |\rho(z_1, c')|$, and the claim follows. $\square_{\text{claim}}$

We are now ready to finish the proof of this lemma. We have already shown that η is finite-to-one, so let $Y''_n \subseteq Y''$, $n \geq 1$, be such that $x \mapsto \eta(x)$ is n-to-one on Y''_n. Let $R_n = \eta(Y''_n)$, and recall that $R_n \subseteq Y$. The sets R_n are pairwise disjoint. Let $\phi_{k,n} : R_n \to Y''_n$, $1 \leq k \leq n$, be Borel bijections that pick the kth point in the preimage: $Y''_n = \bigsqcup_{i=1}^{n} \phi_{k,n}(R_n)$. Note that the maps $\phi_{k,n} : R_n \to \phi_{k,n}(R_n)$ are measure-preserving, since they belong to the pseudo full group of T, and $\sum_{k=1}^{n} |\rho_{T_{\mathcal{R}_C}}(\phi_{k,n}(x))| \leq \beta(x)$ for all $x \in R_n$ by Claim 3. One now has

$$\begin{aligned}\int_{Y''_n} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) &= \sum_{k=1}^{n} \int_{\phi_{k,n}(R_n)} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) \\ \because \phi_{n,k} \text{ are measure-preserving } &= \int_{R_n} \sum_{k=1}^{n} |\rho_{T_{\mathcal{R}_C}}(\phi_{k,n}^{-1}(x))|\, d\mu(x) \\ \because \text{Claim 3 } &\leq \int_{R_n} \beta(x)\, d\mu(x).\end{aligned}$$

Summing these inequalities over n, we get

$$\begin{aligned}\int_{Y''} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) &= \sum_{n=1}^{\infty} \int_{Y''_n} |\rho_{T_{\mathcal{R}_C}}(x)|\, d\mu(x) \\ &\leq \sum_{n=1}^{\infty} \int_{R_n} \beta(x)\, d\mu(x) \leq \int_{Y} \beta(x)\, d\mu(x),\end{aligned}$$

where the last inequality is based on the fact that the sets R_n are pairwise disjoint. This finishes the proof of the inequality (8.2) as well as the lemma. ■

Several important facts follow easily from Lemma 8.3. For one, it implies that for any cross-section C, the intermitted transformation $T_{\mathcal{R}_C}$ belongs to $[\mathbb{R} \curvearrowright X]_1$. In fact, we have the following inequality on the norms.

Corollary 8.4. *For any intermitted transformation $T_{\mathcal{R}_C}$, one has $\|T_{\mathcal{R}_C}\|_1 \leq \|T\|_1$.*

Proof. By the definition of the set Y in Lemma 8.3, $\rho_{T_{\mathcal{R}_C}}(x) = \rho_T(x)$ for all $x \notin Y$, hence

$$\begin{aligned}\int_X |\rho_{T_{\mathcal{R}_C}}|\, d\mu &= \int_{X\setminus Y} |\rho_{T_{\mathcal{R}_C}}|\, d\mu + \int_Y |\rho_{T_{\mathcal{R}_C}}|\, d\mu \\ \because \text{ Lemma 8.3} &\leq \int_{X\setminus Y} |\rho_T|\, d\mu + \int_Y |\rho_T|\, d\mu = \int_X |\rho_T|\, d\mu,\end{aligned}$$

which shows $\|T_{\mathcal{R}_C}\|_1 \leq \|T\|_1$. ■

Remark 8.5. As we discussed in Remark 8.2, the concept of an intermitted transformation applies more broadly than the case of one-dimensional flows. We mention, however, that the analog of Lemma 8.3 and Corollary 8.4 does not hold even for free measure-preserving $\mathbb{R}^2$-flows. Consider an annulus depicted in Figure 8.2a and let T be the rotation by an angle α around the center of this annulus. Let the equivalence relation E consist of two classes, each composing half of the ring. For a point x such that $\neg xETx$, $T_E x$ will be close to the other side of the class. It is easy to arrange the parameters (the angle α and the radii of the annulus) so that $\|\rho_{T_E}(x)\| > \|\rho_T(x)\|$ for all x such that $Tx \neq T_E x$.

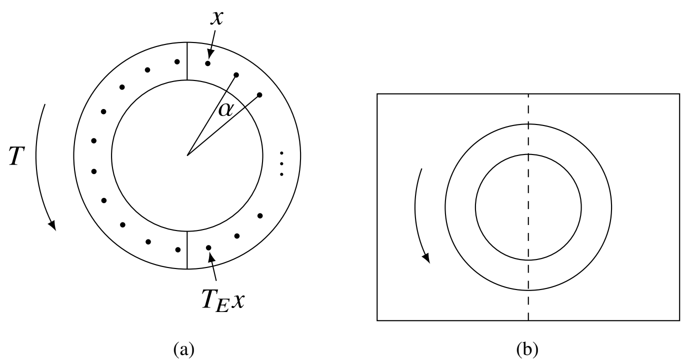

Figure 8.2. Construction of a conservative transformation T with $\|T_E\|_1 > \|T\|_1$.

Every free measure-preserving flow $\mathbb{R}^2 \curvearrowright X$ admits a tiling of its orbits by rectangles. The transformation $T \in [\mathbb{R}^2 \curvearrowright X]_1$ can be defined similarly to Figure 8.2a on each rectangle of the tiling by splitting each tile into two equivalence classes as in Figure 8.2b. The resulting transformation T will have bounded orbits and satisfy $\|T_E\|_1 > \|T\|_1$ relative to the equivalence relation E whose classes are the half tiles.

When the gaps in a cross-section C are large, x and Tx will often be $\mathcal{R}_C$-equivalent, and it is therefore natural to expect that $T_{\mathcal{R}_C}$ will be close to T. This intuition is indeed valid, and the following approximation result is the most important consequence of Lemma 8.3.

Lemma 8.6. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative transformation. For any $\epsilon > 0$, there exists M such that for any cross-section C with $\mathrm{gap}_C(c) \geq M$ for all $c \in C$, one has $\|T \circ T_{\mathcal{R}_C}^{-1}\|_1 < \epsilon$.*

Proof. Let $A_K = \{x \in X : |\rho_T(x)| \geq K\}$, $K \in \mathbb{R}^{\geq 0}$, be the set of points whose cocycle is at least K in absolute value. Since $T \in [\mathbb{R} \curvearrowright X]_1$, we may pick $K \geq 1$ so large that $\int_{A_K} |\rho_T| \, d\mu < \epsilon/4$. Pick any real M such that $2K^2/M < \epsilon/4$. We claim that it satisfies the conclusion of the lemma. To verify this, we pick a cross-section C with all gaps having a size of at least M. Set as before $Y = \{x \in X : Tx \neq T_{\mathcal{R}_C}x\}$. Since

$$\left\|T \circ T_{\mathcal{R}_C}^{-1}\right\|_1 = \int_Y D(Tx, T_{\mathcal{R}_C}x)\, d\mu(x),$$

our task is to estimate this integral. This can be done in a rather crude way. We can simply use the triangle inequality $D(Tx, T_{\mathcal{R}_C}x) \leq |\rho_T(x)| + |\rho_{T_{\mathcal{R}_C}}(x)|$, and deduce

$$\int_Y D(Tx, T_{\mathcal{R}_C}x)\, d\mu(x) \leq \int_Y |\rho_T| \, d\mu + \int_Y |\rho_{T_{\mathcal{R}_C}}| \, d\mu \leq 2 \int_Y |\rho_T| \, d\mu,$$

where the last inequality is based on Lemma 8.3.

It remains to show that $\int_Y |\rho_T| \, d\mu < \epsilon/2$. Let $\widetilde{X} = \{c + [K, \mathrm{gap}_C(c) - K] : c \in C\}$ be the region that leaves out intervals of length K on both sides of each point in C. Note that for any $x \in \widetilde{X} \setminus A_K$ one has $x \mathcal{R}_C Tx$ and thus $T_{\mathcal{R}_C}x = Tx$ for such points. Therefore, $Y \subseteq A_K \sqcup B_K$, where $B_K = X \setminus (\widetilde{X} \cup A_K)$, and thus

$$\int_Y |\rho_T| \, d\mu \leq \int_{A_K} |\rho_T| \, d\mu + \int_{B_K} |\rho_T| \, d\mu < \epsilon/4 + K \cdot 2K/M < \epsilon/2. \qquad \blacksquare$$

Lemma 8.7. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a conservative transformation. For any $\epsilon > 0$ there exists a periodic transformation $P \in [T]$ such that $\|T \circ P^{-1}\|_1 < \epsilon$.*

Proof. By Lemma 8.6, we can find a cocompact cross-section C such that $\|T \circ T_{\mathcal{R}_C}^{-1}\| < \epsilon/2$. Let $\tilde{M}$ be an upper bound for gaps in C. Recall that the cocycle $|\rho_{T_{\mathcal{R}_C}}(x)|$ is uniformly bounded by $\tilde{M}$, and, in fact, the same is true for any element in the full group of $T_{\mathcal{R}_C}$. In particular, we may use Rokhlin's lemma to find a periodic $P \in [T_{\mathcal{R}_C}]$ such that $\|T_{\mathcal{R}_C} \circ P^{-1}\| < \epsilon/2\tilde{M}$, and conclude that $\|T_{\mathcal{R}_C} \circ P^{-1}\|_1 < \epsilon/2$. We therefore have

$$\|T \circ P^{-1}\|_1 \leq \|T \circ T_{\mathcal{R}_C}^{-1}\|_1 + \|T_{\mathcal{R}_C} \circ P^{-1}\|_1 < \epsilon. \qquad \blacksquare$$

Corollary 8.8. *If $T \in [\mathbb{R} \curvearrowright X]_1$ is conservative, then T belongs to the derived L^1 full group $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$. In particular, its index satisfies $\mathcal{I}(T) = 0$.*

Proof. By Lemma 8.7, every conservative transformation $T \in [\mathbb{R} \curvearrowright X]_1$ lies in the closed subgroup generated by the periodic elements. This subgroup is equal to the derived L^1 full group $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$ by Corollary 3.16. Since the range of $\mathcal{I}$ is abelian, its kernel contains all commutators, and thus $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1) \subseteq \ker \mathcal{I}$. $\blacksquare$

Chapter 9

Dissipative and monotone transformations

The previous chapter studied conservative transformations, whereas this one concentrates on dissipative ones. Our goal will be to show that any dissipative $T \in [\mathbb{R} \curvearrowright X]_1$ of index $\mathcal{I}(T) = 0$ belongs to the derived subgroup $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$. Recall that conversely, *every* element of the (topological) derived group $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$ has index zero since the index map is a continuous group homomorphism taking values in an abelian topological group. We begin by describing some general aspects of the dynamics of dissipative automorphisms.

Recall that according to Proposition 4.16, any transformation $T \in [\mathbb{R} \curvearrowright X]$ induces a T-invariant partition of the phase space $X = C_T \sqcup D_T$ such that $T|_{C_T}$ is conservative and $T|_{D_T}$ is dissipative. Formally speaking, a transformation is said to be dissipative if the partition trivializes to $D_T = X$. For the purpose of this chapter, it is, however, convenient to widen this notion just a bit by allowing T to have fixed points.

Definition 9.1. A transformation $T \in [\mathbb{R} \curvearrowright X]$ is said to be **dissipatively supported** if $D_T = \operatorname{supp} T$, where D_T is the dissipative element of the Hopf decomposition for T.

9.1 Orbit limits and monotone transformations

We begin by showing that the dynamics of dissipatively supported transformations in L^1 full groups of $\mathbb{R}$-flows is similar to those in L^1 full groups of $\mathbb{Z}$-actions. We do so by establishing an analog of R. M. Belinskaja's result [8, Thm. 3.2]. Recall that a sequence of reals is said to have an almost constant sign if all but finitely many elements of the sequence have the same sign.

Proposition 9.2. *Let S be a measure-preserving transformation of the real line that commensurates the set $\mathbb{R}^-$. Suppose that S is dissipatively supported. Then for almost all $x \in \mathbb{R}$, the sequence of reals $(S^k(x) - x)_{k\in\mathbb{N}}$ has an almost constant sign.*

Proof. Let Q be the set of reals x such that the sequence $(S^k(x) - x)_{k\in\mathbb{N}}$ does not have an almost constant sign. Assume, to the contrary, that Q has positive measure. Since S is dissipative, we can find a Borel wandering set $A \subseteq \mathbb{R}$ for S that intersects Q non-trivially. All the translates of $Q' = Q \cap A$ are disjoint, and for all $x \in Q'$, the sequence $(S^k(x) - x)_{k\in\mathbb{N}}$ does not have an almost constant sign.

Since S is dissipatively supported, for almost all $x \in Q'$, the sequence of absolute values $(|S^k(x)|)_{k\in\mathbb{N}}$ tends to $+\infty$ (see Proposition C.4). In particular, there are infinitely many points y in the S-orbit of x such that $y < 0$ but $S(y) > 0$. Because the map

$Q' \times \mathbb{Z} \to \mathbb{R}$, which sends (x, k) to $S^k(x)$, is measure-preserving, it follows that the set of $y < 0$ such that $S(y) > 0$ has infinite measure. This contradicts the fact that S commensurates the set $\mathbb{R}^-$. ■

Corollary 9.3. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be dissipatively supported. For almost all $x \in \operatorname{supp} T$, the sequence $(\rho(x, T^k(x)))_{k\in\mathbb{N}}$ has an almost constant sign.*

Proof. Let $T \in [\mathbb{R} \curvearrowright X]_1$. For all $x \in X$, denote by T_x the measure-preserving transformation of $\mathbb{R}$ induced by T on the $\mathbb{R}$-orbit of x. By the proof of Proposition 6.8, the integral

$$\int_X \lambda(\mathbb{R}^{\geq 0} \vartriangle (T_x(\mathbb{R}^{\geq 0})))\, d\mu(x)$$

is finite. In particular, for almost every $x \in X$, the transformation T_x commensurates the set $\mathbb{R}^{\geq 0}$. The conclusion now follows directly from the previous proposition. ■

For any dissipatively supported transformation in an L^1 full group of a free locally compact Polish group action and for almost every $x \in X$, $\rho(x, T^n x) \to \infty$ as $n \to \infty$, in the sense that $\rho(x, T^n x)$ eventually escapes any compact subset of the acting group. In the context of flows, Corollary 9.3 strengthens this statement and implies that $\rho(x, T^n x)$ must converge to either $+\infty$ or $-\infty$.

Corollary 9.4. *If $T \in [\mathbb{R} \curvearrowright X]_1$ is dissipatively supported, then for almost every point $x \in \operatorname{supp} T$, either $\lim_{n\to\infty} \rho(x, T^n x) = +\infty$ or $\lim_{n\to\infty} \rho(x, T^n x) = -\infty$.* ■

In view of this corollary, there is a canonical T-invariant decomposition of $\operatorname{supp} T$ into "positive" and "negative" orbits.

Definition 9.5. Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipatively supported automorphism. Its support is partitioned into $\vec{X} \sqcup \overleftarrow{X}$, where

$$\vec{X} = \left\{x \in \operatorname{supp} T : \lim_{n\to\infty} \rho(x, T^n x) = +\infty\right\},$$
$$\overleftarrow{X} = \left\{x \in \operatorname{supp} T : \lim_{n\to\infty} \rho(x, T^n x) = -\infty\right\}.$$

The set $\vec{X}$ is said to be **positive evasive**, and $\overleftarrow{X}$ is **negative evasive**.

According to Corollary 9.3, for almost every $x \in \operatorname{supp} T$, *eventually* either all $T^n x$ are to the right of x or all are to the left of it. There are points x for which the adverb "eventually" can, in fact, be dropped.

Corollary 9.6. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipatively supported transformation, and let*

$$\vec{A} = \{x \in \vec{X} : \rho(x, T^n x) > 0 \text{ for all } n \geq 1\},$$
$$\overleftarrow{A} = \{x \in \overleftarrow{X} : \rho(x, T^n x) < 0 \text{ for all } n \geq 1\}.$$

The set $A = \vec{A} \sqcup \overleftarrow{A}$ is a complete section for $T|_{\operatorname{supp} T}$.

Proof. We need to show that almost every orbit of T intersects A. Let $x \in \operatorname{supp} T$ and suppose for definiteness that $x \in \vec{X}$. Since $\lim_{n\to\infty} \rho(x, T^n x) = +\infty$, we can define $n_0 = \max\{n \in \mathbb{N} : \rho(x, T^n x) \leq 0\}$, and then $T^{n_0} x \in \vec{A}$. ∎

Definition 9.7. A dissipatively supported transformation $T \in [\mathbb{R} \curvearrowright X]_1$ is **monotone** if $\rho(x, Tx) > 0$ for almost all $x \in \vec{X}$, and $\rho(x, Tx) < 0$ for almost all $x \in \overleftarrow{X}$.

Corollary 9.8. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipatively supported transformation. There is a complete section $A \subseteq \operatorname{supp} T$ and a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1 \cap [T]$ such that $T = P \circ T_A$ and T_A is monotone.*

Proof. Take A to be as in Corollary 9.6 and note that $P = T \circ T_A^{-1}$ is periodic and satisfies the conclusions of the corollary. ∎

As we discussed at the beginning of the chapter, our goal is to show that the kernel of the index map coincides with the derived subgroup of $[\mathbb{R} \curvearrowright X]_1$. Note that if $T = P \circ T_A$ is as above, then $\mathcal{I}(T) = \mathcal{I}(T_A)$, and coupled with the results of Chapter 8, it will suffice to show that all monotone transformations of index zero belong to $\mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$. This will be the focus of the rest of this chapter and will take some effort to achieve, but the main strategy is to show that such transformations can be approximated by periodic transformations, which is the content of Theorem 9.16 below.

9.2 Arrival and departure sets

Throughout the rest of this chapter, we fix a cross-section $C \subset X$ and a monotone transformation $T \in [\mathbb{R} \curvearrowright X]_1$.

Let us recall a few definitions and facts from Section 1.2.4. The lacunarity of C provides the gap function $\operatorname{gap}_C : C \to (0, +\infty)$, and the induced map $\sigma_C : C \to C$ taking c to $c + \operatorname{gap}_C(c)$, whose orbits are the intersections of the flow's orbits with C. We let λ_c^C be the Lebesgue measure on $c + [0, \operatorname{gap}_C(c))$ given by

$$\lambda_c^C(A) = \lambda(\{t \in [0, \operatorname{gap}_C(c)) : c + t \in A\}).$$

The measure μ can be disintegrated as $\mu(A) = \int_C \lambda_c^C(A)\, d\nu(c)$ for some finite (but not necessarily probability) measure ν on C. Finally, we use the convenient notation $A(c) = A \cap \big(c + [0, \operatorname{gap}_C(c))\big) = A \cap [c]_{\mathcal{R}_C}$. Note that $\lambda_c(A(c)) = \lambda_c^C(A)$ for all $c \in C$, where λ_c denotes the Lebesgue measure on the whole orbit of c.

We now introduce some essential additional terminology concerning our fixed monotone transformation $T \in [\mathbb{R} \curvearrowright X]_1$. The **arrival set** A_C is the set of the first visitors to $\mathcal{R}_C$ classes: $A_C = \{x \in \operatorname{supp} T : \neg x \mathcal{R}_C T^{-1} x\}$. Analogously, the **departure set** D_C is defined to be $D_C = \{x \in \operatorname{supp} T : \neg x \mathcal{R}_C T x\}$. We also let $\vec{A}_C$ denote $A_C \cap \vec{X}$ and

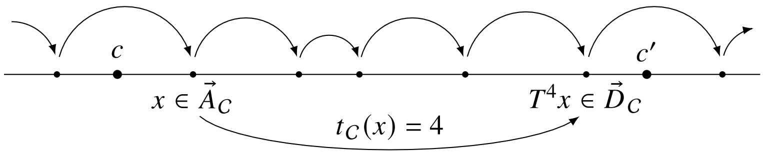

Figure 9.1. Arrival and departure sets.

$\overleftarrow{A}_C = A_C \cap \overleftarrow{X}$; likewise for $\vec{D}_C$ and $\overleftarrow{D}_C$. Note that $T(D_C) = A_C$, and thus $T^{-1}(A_C) = D_C$. There is, however, another useful map from A_C onto D_C.

We define the **transfer value** $t_C : A_C \to \mathbb{N}$ by the condition

$$t_C(x) = \min\{n \geq 0 : T^n x \in D_C\}$$

and the **transfer function** $\tau_C : A_C \to D_C$ is defined to be $\tau_C(x) = T^{t_C(x)}x$. Note that τ_C is measure-preserving. The transfer value introduces a partition of the arrival set $A_C = \bigsqcup_{n\in\mathbb{N}} A_C^n$, where $A_C^n = t_C^{-1}(n)$. By applying the transfer function, we also obtain a partition for the departure set: $D_C = \bigsqcup_{n\in\mathbb{N}} D_C^n$, where $D_C^n = \tau_C(A_C^n)$.

In plain words, $t_C(x) + 1$ is the number of points in $[x]_{\mathcal{R}_T} \cap [x]_{\mathcal{R}_C}$. Therefore if $\lambda_c^C(A_C^n) \geq \lambda_c^C(A_C^m)$ for some $n \geq m$ then also $\lambda_c^C([A_C^n]_{\mathcal{R}_T}) \geq \lambda_c^C([A_C^m]_{\mathcal{R}_T})$ since

$$\lambda_c^C([A_C^n]_{\mathcal{R}_T}) = (n+1)\lambda_c^C(A_C^n) \geq (m+1)\lambda_c^C(A_C^m) = \lambda_c^C([A_C^m]_{\mathcal{R}_T}).$$

In Sections 9.3 and 9.4, we modify the transformation T on the arrival and departure sets, and we want to do this in a way that affects as many orbits as possible, as measured by λ_c^C. This amounts to using sets A_C^n (and D_C^n) with as high values of n as possible. The next lemma will be helpful in conducting such a selection in a measurable way across all of $c \in C$.

Lemma 9.9. *Let $A \subseteq X$ be a measurable set with a measurable partition $A = \bigsqcup_n A_n$ and let $\xi : C \to \mathbb{R}^{\geq 0}$ be a measurable function such that $\xi(c) \leq \lambda_c^C(A)$ for all $c \in C$. There are measurable $\nu : C \to \mathbb{N}$ and $r : C \to \mathbb{R}^{\geq 0}$ such that for any $c \in C$ for which $\xi(c) > 0$, one has*

$$\lambda_c^C\Big(\Big(\bigsqcup_{n>\nu(c)} A_n\Big) \cup \big(A_{\nu(c)} \cap (c + [0, r(c)])\big)\Big) = \xi(c). \tag{9.1}$$

Proof. For $c \in C$ such that $\xi(c) > 0$, set

$$\nu(c) = \min\Big\{n \in \mathbb{N} : \lambda_c^C\Big(\bigsqcup_{k>n} A_k\Big) < \xi(c)\Big\}.$$

Note that one necessarily has $\lambda_c^C(A_{\nu(c)}) \geq \xi(c) - \lambda_c^C\big(\bigsqcup_{n>\nu(c)} A_n\big) > 0$. Set

$$r(c) = \min\Big\{a \geq 0 : \lambda_c^C\big(A_{\nu(c)} \cap (c + [0, a])\big) = \xi(c) - \lambda_c^C\Big(\bigsqcup_{n>\nu(c)} A_n\Big)\Big\}.$$

These functions ν and r satisfy the conclusions of the lemma. ∎

Remark 9.10. Note that Eq. (9.1) does not specify the functions ν and r uniquely. For instance, although $\lambda_c^C(A_{\nu(c)}) > 0$, there might be $\delta > 0$ such that, for some $c \in C$,

$$\lambda_c^C\big(A_{\nu(c)} \cap (c + [r(c), r(c) + \delta])\big) = 0.$$

In this case, replacing $r(c)$ with $r(c) + \delta$ does not change the validity of Eq. (9.1).

Definition 9.11. Consider the partition of the positive arrival set $\vec{A}_C = \bigsqcup_n \vec{A}_C^n$ and let $\xi : C \to \mathbb{R}^{\geq 0}$, $r : C \to \mathbb{R}^{\geq 0}$, and $\nu : C \to \mathbb{N}$ be as in Lemma 9.9. The set $\vec{A}_C^\xi$ defined by the condition

$$\vec{A}_C^\xi(c) = \bigsqcup_{n > \vec{\nu}(c)} \vec{A}_C^n(c) \cup \big(A_C^{\vec{\nu}(c)} \cap (c + [0, \vec{r}(c)])\big) \quad \text{for all } c \in C$$

is said to be the **positive ξ-copious arrival set**. The **positive ξ-copious departure set** is given by $\vec{D}_C^\xi = \tau_C(\vec{A}_C^\xi)$. The definitions of the **negative ξ-copious arrival** and **departure sets** use the partition $\overleftarrow{A}_C = \bigsqcup_n \overleftarrow{A}_C^n$ of the negative arrival set and are analogous.

Copious sets maximize the measure λ_c^C of their saturation under the action of T. In other words, among all subsets $A' \subseteq \vec{A}_C$ for which $\lambda_c^C(A') = \xi(c)$, the measure $\lambda_c^C([A']_{\mathcal{R}_T})$ is maximal when $A'(c) = \vec{A}_C^\xi(c)$. In particular, if $\lambda_c^C(\vec{A}_C^\xi)$ is close to $\lambda_c^C(\vec{A}_C)$, then we expect $\lambda_c^C([\vec{A}_C^\xi]_{\mathcal{R}_T})$ to be close to $\lambda_c^C([\vec{A}_C]_{\mathcal{R}_T})$. The following lemma quantifies this intuition.

Lemma 9.12. *Let $\xi : C \to \mathbb{R}^{\geq 0}$ be such that $\xi(c) \leq \lambda_c^C(\vec{A}_C)$ for all $c \in C$, and let $\vec{A}_C^\xi$ be the ξ-copious arrival set constructed in Lemma 9.9. If there exists $1/2 > \delta > 0$ such that $\xi(c) \geq (1 - \delta)\lambda_c^C(\vec{A}_C)$ for all $c \in C$, then*

$$\lambda_c^C([\vec{A}_C(c) \setminus \vec{A}_C^\xi(c)]_{\mathcal{R}_T}) \leq \frac{\delta}{1 - \delta}\lambda_c^C(\vec{X}) \quad \textit{for all } c \in C,$$

and therefore also $\mu([\vec{A}_C \setminus \vec{A}_C^\xi]_{\mathcal{R}_T}) \leq \frac{\delta}{1-\delta}\mu(\vec{X})$.

An analogous statement is valid for the negative arrival set $\overleftarrow{A}_C$.

Proof. Let ν be as in Lemma 9.9 and note that

$$\bigsqcup_{k > \nu(c)} \vec{A}_C^k(c) \subseteq \vec{A}_C^\xi(c) \subseteq \bigsqcup_{k \geq \nu(c)} \vec{A}_C^k(c)$$

whenever $c \in C$ satisfies $\xi(c) > 0$. Recall that for $x \in \vec{A}^n_C$ we have $x\mathcal{R}_C T^k$ for all $0 \le k \le n$ and the sets $T^k(\vec{A}^n_C)$ are pairwise disjoint. In particular,

$$\begin{aligned}\lambda^C_c(\vec{X}) \ge \lambda^C_c\Big(\Big[\bigsqcup_{k \ge \nu(c)} \vec{A}^k_C(c)\Big]_{\mathcal{R}_T}\Big) &\ge (\nu(c)+1)\lambda^C_c\Big(\bigsqcup_{k\ge\nu(c)} \vec{A}^k_C(c)\Big)\\ &\ge (\nu(c)+1)\lambda^C_c(\vec{A}^\xi_C) = (\nu(c)+1)\xi(c). \end{aligned} \tag{9.2}$$

Note also that $\xi(c) \ge (1-\delta)\lambda^C_c(\vec{A}_C)$ implies

$$\lambda^C_c(\vec{A}_C \setminus \vec{A}^\xi_C) \le \xi(c)\delta/(1-\delta). \tag{9.3}$$

For any $c \in C$, we have

$$\begin{aligned}\lambda^C_c([\vec{A}_C(c) \setminus \vec{A}^\xi_C(c)]_{\mathcal{R}_T}) &\le \lambda^C_c(\{T^k x : x \in \vec{A}_C(c) \setminus \vec{A}^\xi_C(c), 0 \le k \le \vec{\nu}(c)\})\\ &\le (\vec{\nu}(c)+1)\lambda^C_c(\vec{A}_C \setminus \vec{A}^\xi_C)\\ \because\ (9.3)\ &\le (\vec{\nu}(c)+1)\vec{\xi}(c)\delta/(1-\delta)\\ \because\ (9.2)\ &\le \lambda^C_c(\vec{X})\delta/(1-\delta).\end{aligned}$$

The inequality for the measure μ follows by disintegrating μ into $\int_C \lambda^C_c(\,\cdot\,)\,d\nu(c)$, as discussed at the outset of this section.

The argument for the negative arrival set is completely analogous. ■

9.3 Coherent modifications

We remind the reader that our goal is to show that any dissipatively supported transformation $T \in [\mathbb{R} \curvearrowright X]_1$ of index $\mathcal{I}(T) = 0$ can be approximated by periodic transformations. One approach to "loop" the orbits of T is by mapping $\vec{D}_C(c)$ to $\overleftarrow{A}_C(c)$ and $\overleftarrow{D}_C(c)$ to $\vec{A}_C(c)$ (cf. Figure 9.6). For such a modification to work, the measures $\lambda^C_c(\vec{D}_C(c))$ and $\lambda^C_c(\overleftarrow{A}_C(c))$ have to be equal. Recall that $\mathcal{I}(T) = 0$ implies that for almost every $c \in C$, the measure of points x such that $x \le c < Tx$ equals the measure of those y for which $Ty < c \le y$. If one could guarantee that $T(\vec{D}_C(c)) = \vec{A}_C(\sigma_C(c))$, then the aforementioned modification would indeed work. In the case of $\mathbb{Z}$-actions, the discreteness of the acting group allows one to find a cross-section C for which this condition does hold. For flows, however, we must deal with the possibility that $T(\vec{D}_C(c))$ can be "scattered" (see Figure 9.4) along the orbit and be unbounded, which is the key reason for the increased complexity compared to the argument for $\mathbb{Z}$-actions.

Since we cannot hope to "loop" all the orbits of T, we will do the next best thing, and apply the modification of Figure 9.6 on "most" orbits as measured by λ^C_c. Copious sets discussed in Section 9.2 have large saturations under T, but, generally speaking, fail to satisfy $T(\vec{D}^\xi_C(c)) = \vec{A}^\xi_C(\sigma_C(c))$ for the same reason as do the sets $\vec{D}_C(c)$. Our strategy is to leverage the "ϵ of room" provided by the difference $\vec{D}_C(c) \setminus \vec{D}_C^{\xi}(c)$ to transform T into T' that will retain the same arrival and departure sets as T, while additionally satisfying the condition $T'(\vec{D}_C^{\xi}(c)) = \vec{A}_C^{\xi}(\sigma_C(c))$. In this section, we describe two abstract modifications of dissipatively supported transformations, and the approximation strategy outlined above will later be implemented in Section 9.4.

Since we are about to consider arrival and departure sets of different transformations, we use the notation $\vec{A}_C[U]$ to denote the positive arrival set constructed for a transformation U; likewise for negative arrival and departure sets, etc.

Lemma 9.13. *Let ϕ and ϕ' be measure-preserving transformations on X subject to the following conditions:*

(1) $\operatorname{supp}(\phi) \subseteq D_C$, $\operatorname{supp}(\phi') \subseteq A_C$;
(2) $\phi(\vec{D}_C) = \vec{D}_C$, $\phi(\overleftarrow{D}_C) = \overleftarrow{D}_C$, *and* $\phi'(\vec{A}_C) = \vec{A}_C$, $\phi'(\overleftarrow{A}_C) = \overleftarrow{A}_C$;
(3) $x \mathcal{R}_C \phi(x)$ *and* $x \mathcal{R}_C \phi'(x)$ *for all* $x \in \operatorname{supp} T$.

The transformation $Ux = \phi' T \phi(x)$ is monotone, $Ux = Tx$ for all $x \notin D_C$, and the sets D_C, A_C remain the same:

$$
\begin{aligned}
\vec{X}[U] &= \vec{X} & \overleftarrow{X}[U] &= \overleftarrow{X}, \\
\vec{D}_C[U] &= \vec{D}_C & \overleftarrow{D}_C[U] &= \overleftarrow{D}_C, \\
\vec{A}_C[U] &= \vec{A}_C & \overleftarrow{A}_C[U] &= \overleftarrow{A}_C.
\end{aligned}
$$

Moreover, the integral of lengths of "departing arcs" remains unchanged:

$$\int_{D_C} |\rho_U| \, d\mu = \int_{D_C} |\rho_T| \, d\mu,$$

and the following estimate on $\int_X D(Tx, Ux) \, d\mu(x)$ is available:

$$\int_X D(Tx, Ux) \, d\mu(x) \le 2 \int_{D_C} |\rho_T(x)| \, d\mu(x).$$

Proof. Figure 9.2 illustrates the definition of the transformation U. The equality of the arrival and departure sets is straightforward to verify. Note that $\phi(\vec{D}_C(c)) = \vec{D}_C(c)$ for all $c \in C$, and therefore $\int_{\vec{D}_C} \rho_\phi \, d\mu = 0$. In fact, the following four integrals vanish:

$$\int_{\vec{D}_C} \rho_\phi \, d\mu = \int_{\overleftarrow{D}_C} \rho_\phi \, d\mu = \int_{\vec{A}_C} \rho_{\phi'} \, d\mu = \int_{\overleftarrow{A}_C} \rho_{\phi'} \, d\mu = 0. \tag{9.4}$$

Observe that ρ_U is positive on $\vec{D}_C$ and negative on $\overleftarrow{D}_C$; thus

$$\int_{D_C} |\rho_U| \, d\mu = \int_{\vec{D}_C} \rho_{\phi' T \phi} \, d\mu - \int_{\overleftarrow{D}_C} \rho_{\phi' T \phi} \, d\mu$$

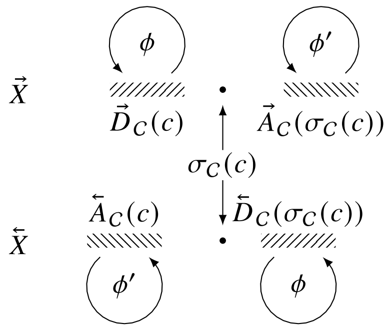

Figure 9.2. The transformation $U = \phi' T \phi$ defined in Lemma 9.13.

$$
\begin{aligned}
\because \text{ cocycle identity} &= \int_{\vec{D}_C} \rho_\phi \, d\mu + \int_{\vec{D}_C} \rho_T(\phi(x)) \, d\mu(x) + \int_{\vec{D}_C} \rho_{\phi'}(T\phi(x)) \, d\mu(x) \\
&\quad - \int_{\overleftarrow{D}_C} \rho_\phi \, d\mu - \int_{\overleftarrow{D}_C} \rho_T(\phi(x)) \, d\mu(x) - \int_{\overleftarrow{D}_C} \rho_{\phi'}(T\phi(x)) \, d\mu(x) \\
&= \int_{\vec{D}_C} \rho_\phi \, d\mu + \int_{\vec{D}_C} \rho_T \, d\mu + \int_{\vec{A}_C} \rho_{\phi'} \, d\mu \\
&\quad - \int_{\overleftarrow{D}_C} \rho_\phi \, d\mu - \int_{\overleftarrow{D}_C} \rho_T \, d\mu - \int_{\overleftarrow{A}_C} \rho_{\phi'} \, d\mu \\
\because \text{ Eq. (9.4)} &= \int_{\vec{D}_C} \rho_T \, d\mu - \int_{\overleftarrow{D}_C} \rho_T \, d\mu = \int_{D_C} |\rho_T| \, d\mu.
\end{aligned}
$$

Finally, note that for any $x \in D_C$, the arc from x to Tx intersects the arc from $T^{-1}\phi' T\phi(x)$ to $\phi' T \phi(x)$ (both arcs go over the same point of C), and therefore

$$D(Tx, Ux) \leq |\rho_T(x)| + |\rho_T(T^{-1}\phi' T\phi(x))|.$$

Integration over D_C yields

$$\int_X D(Tx, Ux) \, d\mu(x) = \int_{D_C} D(Tx, Ux) \, d\mu(x) \leq 2 \int_{D_C} |\rho_T(x)| \, d\mu(x). \qquad \blacksquare$$

Lemma 9.14. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation. Let $F \subseteq D_C$ be such that $\lambda_c^C(\vec{F}) = \lambda_c^C(\overleftarrow{F})$ for all $c \in C$, and the function $C \ni c \mapsto \lambda_c^C(F)$ is σ_C-invariant (i.e., $\lambda_c^C(F) = \lambda_{c'}^C(F)$ whenever c and c' belong to the same orbit of the flow). Let $Z \subseteq A_C$ be the arrival subset that corresponds to F, i.e., $Z = T(F)$. Let $\psi : \vec{F} \to \overleftarrow{Z}$ and $\psi' : \overleftarrow{F} \to \vec{Z}$ be any measure-preserving transformations such that $\psi(x) \mathcal{R}_C x$ and $\psi'(x) \mathcal{R}_C x$ for* *all x in the corresponding domains. Define $V : X \to X$ by the following formula:*

$$Vx = \begin{cases} \psi(x) & \text{if } x \in \vec{F}, \\ \psi'(x) & \text{if } x \in \overleftarrow{F}, \\ Tx & \text{otherwise.} \end{cases}$$

The transformation V is a measure-preserving automorphism from the full group $[\mathbb{R} \curvearrowright X]$, and $Vx = Tx$ for all $x \notin F$. The integral of distances $D(Tx, Vx)$ can be estimated as follows:

$$\int_X D(Tx, Vx)\, d\mu(x) \leq 2 \int_{D_C} |\rho_T(x)|\, d\mu(x).$$

The following figure illustrates the notions of Lemma 9.14.

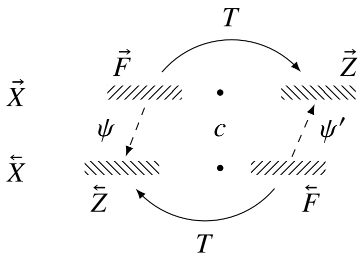

Figure 9.3. The transformation V defined in Lemma 9.14.

Proof. It is straightforward to verify that V is a measure-preserving transformation. For the integral inequality, note that for any $x \in \vec{F}$ one has

$$D(Tx, Vx) \leq |\rho_T(x)| + |\rho_T(T^{-1}x)|,$$

and therefore

$$\int_{\vec{F}} D(Tx, Vx)\, d\mu(x) \leq \int_{\vec{F}} |\rho_T|\, d\mu + \int_{\overleftarrow{F}} |\rho_T|\, d\mu = \int_F |\rho_T|\, d\mu \leq \int_{D_C} |\rho_T|\, d\mu.$$

A similar inequality holds for $\int_{\overleftarrow{F}} D(Tx, Vx)\, d\mu$, and the lemma follows. ∎

9.4 Periodic approximations

We now have all the ingredients necessary to prove that monotone transformations can be approximated by periodic automorphisms. Our arguments follow the approach outlined at the beginning of Section 9.3.

In the following lemma, we assume that the Lebesgue measure of those $x \in \vec{X}$ that jump over any given $c \in C$ is bounded from above by some β, and that most of such jumps — of measure at least γ — are between adjacent $\mathcal{R}_C$-classes. We are going to construct a periodic approximation P of the transformation T with an explicit bound on $\int_X D(Tx, Px)\, d\mu(x)$, which can be made small for a sufficiently sparse cross-section C. When the flow is ergodic, this lemma alone suffices to conclude that $T \in \mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$. Theorem 9.16 builds upon Lemma 9.15 and treats the general case.

Lemma 9.15. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation, let $K > 0$ be a positive real, and let $J = \{x \in \operatorname{supp} T : |\rho_T(x)| \geq K\}$. Let C be a cross-section such that $\operatorname{gap}_C(c) > K$ for all $c \in C$. Let $0 < \gamma < \beta$ be reals such that for all $c \in C$:*

$$\lambda_c^C(\{x \in \vec{X} : x < \sigma_C(c) \leq Tx, Tx\, \mathcal{R}_C\, \sigma_C(c)\}) > \gamma,$$
$$\lambda_c^C(\{x \in \overleftarrow{X} : Tx < c \leq x, Tx\, \mathcal{R}_C\, \sigma_C^{-1}(c)\}) > \gamma,$$
$$\lambda_c^C(\{x \in \vec{X} : x < \sigma_C(c) \leq Tx\}) < \beta,$$
$$\lambda_c^C(\{x \in \overleftarrow{X} : Tx < c \leq x\}) < \beta.$$

There exists a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1$ such that $\operatorname{supp} P \subseteq \operatorname{supp} T$ and

$$\int_X D(Tx, Px)\, d\mu(x) \leq 5 \int_{D_C} |\rho_T|\, d\mu + \int_J |\rho_T|\, d\mu + \frac{K(\beta - \gamma)}{\gamma} \mu(\operatorname{supp} T).$$

Proof. Let D_C and A_C be the departure and arrival sets of T. Figure 9.4 depicts the arrival set $\vec{A}_C(c)$ and the departure set $\vec{D}_C(c)$ for an element c of the cross-section C. Note that the preimages $T^{-1}(\vec{A}_C(c))$ may come from different (possibly infinitely many) $\mathcal{R}_C$-equivalence classes; likewise, the images $T(\vec{D}_C(c))$ of the departure set may visit several $\mathcal{R}_C$-equivalence classes.

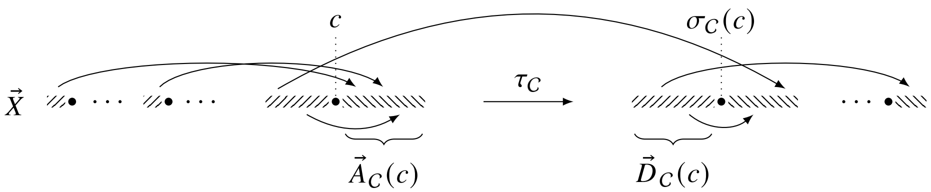

Figure 9.4. The arrival set $\vec{A}_C(c)$ and the departure set $\vec{D}_C(c)$ for some $c \in C$.

Set $\xi(c) = \gamma$ to be the constant function. In view of the assumptions on γ, we may apply Lemma 9.9 to get positive and negative ξ-copious arrival sets $\vec{A}_C^{\xi} \subseteq A_C$ and $\overleftarrow{A}_C^{\xi} \subseteq A_C$, as well as the corresponding departure sets $\vec{D}_C^{\xi} = \tau_C(\vec{A}_C^{\xi})$ and $\overleftarrow{D}_C^{\xi} = \tau_C(\overleftarrow{A}_C^{\xi})$. For the sets $A_C^\xi = \vec{A}_C^\xi \sqcup \overleftarrow{A}_C^\xi$ and $D_C^\xi = \vec{D}_C^\xi \sqcup \overleftarrow{D}_C^\xi$, we have $\lambda_c^C(A_C^\xi(c)) = 2\gamma = \lambda_c^C(D_C^\xi(c))$ for all $c \in C$. Let

$$
\begin{aligned}
A_C^\circ &= \left\{x \in \vec{A}_C : T^{-1}x \,\mathcal{R}_C\, \sigma_C^{-1}(\pi_C(x))\right\} \cup \left\{x \in \overleftarrow{A}_C : T^{-1}x \,\mathcal{R}_C\, \sigma_C(\pi_C(x))\right\},\\
D_C^\circ &= \left\{x \in \vec{D}_C : Tx \,\mathcal{R}_C\, \sigma_C(\pi_C(x))\right\} \cup \left\{x \in \overleftarrow{D}_C : Tx \,\mathcal{R}_C\, \sigma_C^{-1}(\pi_C(x))\right\},
\end{aligned}
$$

be the set of arcs that jump from/to *the next* $\mathcal{R}_C$-equivalence class. By the assumptions of the lemma, we have $\lambda_c^C(\vec{D}_C^\circ(c)) \geq \gamma$ and $\lambda_c^C(\vec{A}_C^\circ(c)) \geq \gamma$ for all $c \in C$. Let ϕ be any measure-preserving transformation such that:

- ϕ is supported on D_C;
- $\phi(\vec{D}_C) = \vec{D}_C$ and $\phi(\overleftarrow{D}_C) = \overleftarrow{D}_C$;
- $\phi(x) \,\mathcal{R}_C\, x$ for all $x \in X$;

and moreover

$$\phi(D_C^\xi) \subseteq D_C^\circ. \tag{9.5}$$

Select a transformation ϕ' such that

- ϕ' is supported on A_C;
- $\phi'(\vec{A}_C) = \vec{A}_C$ and $\phi'(\overleftarrow{A}_C) = \overleftarrow{A}_C$;
- $\phi'(x) \,\mathcal{R}_C\, x$ for all $x \in X$;

and moreover

$$\phi'(T \circ \phi(D_C^\xi)) = A_C^\xi. \tag{9.6}$$

Figure 9.5 illustrates these maps. Note that while in general $\tau_C\big(\vec{A}^\circ(c)\big) \neq \vec{D}^\circ(c)$, one has $\tau_C\big(\vec{A}^\xi(c)\big) = \vec{D}^\xi(c)$ for all $c \in C$ by the definition of the ξ-copious departure set.

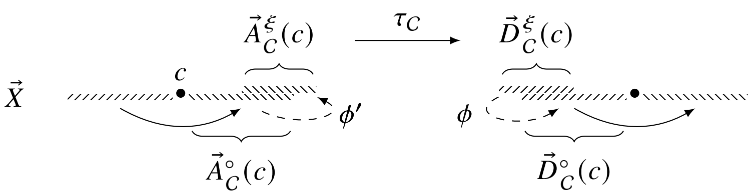

Figure 9.5. Automorphism ϕ maps $D_C^\xi(c)$ into $D_C^\circ(c)$ and $(\phi')^{-1}$ sends $A_C^\xi(c)$ into $A_C^\circ(c)$.

Let U be the transformation obtained by applying Lemma 9.13 to T, ϕ, and ϕ'. The automorphism U satisfies $U(\vec{D}_C^\xi(c)) = \vec{A}_C^\xi(\sigma_C(c))$ and $U(\overleftarrow{D}_C^\xi(c)) = \overleftarrow{A}_C^\xi(\sigma_C^{-1}(c))$ for all $c \in C$. Choose a measure-preserving transformation $\psi : \vec{D}_C^\xi \to \overleftarrow{A}_C^\xi$ such that $x \,\mathcal{R}_C\, \psi(x)$ for all x in the domain of ψ. Set $\psi' = \tau_C^{-1} \circ \psi^{-1} \circ \tau_C^{-1} : \overleftarrow{D}_C^\xi \to \vec{A}_C^\xi$. Let V be the transformation produced by Lemma 9.14 applied to U, ψ, and ψ' (see Figure 9.6). Finally, set $P : X \to X$ to be

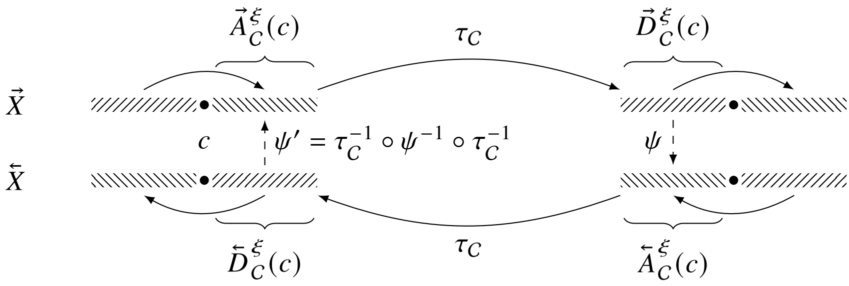

Figure 9.6. Construction of the automorphism V from U, ψ, and ψ'.

$$Px = \begin{cases} Vx & \text{if } x \in [D^{\xi}_C]_{\mathcal{R}_V}, \\ x & \text{otherwise.} \end{cases}$$

We claim that P satisfies the conclusions of the lemma. It is periodic, since the transformation $\psi' \circ \tau_C \circ \psi \circ \tau_C$ is the identity map, and $\operatorname{supp} P \subseteq \operatorname{supp} T$ by construction. It remains to estimate $\int_X D(Tx, Px)\, d\mu(x)$.

$$\begin{aligned} \int_X D(Tx, Px)\, d\mu(x) &\leq \int_X D(Tx, Ux)\, d\mu(x) + \int_X D(Ux, Vx)\, d\mu(x) \\ &\quad + \int_X D(Vx, Px)\, d\mu(x) \\ &\leq [\text{Estimates of Lemma 9.13 and Lemma 9.14}] \\ &\leq 4 \int_{D_C} |\rho_T|\, d\mu + \int_X D(Vx, Px)\, d\mu(x). \end{aligned}$$

We concentrate on estimating $\int_X D(Vx, Px)\, d\mu(x)$. Recall that $Tx = Ux = Vx$ for all $x \notin D_C$; hence, $\rho_T(x) = \rho_V(x)$ for $x \notin D_C$. Set $\Psi = (\vec{X} \cup \overleftarrow{X}) \setminus [D^{\xi}_C]_{\mathcal{R}_V}$ and note that $Vx = Ux$ for $x \in \Psi$. Therefore, using the conclusion of Lemma 9.13, we have

$$\int_{D_C \cap \Psi} |\rho_V|\, d\mu = \int_{D_C \cap \Psi} |\rho_U|\, d\mu \leq \int_{D_C} |\rho_U|\, d\mu = \int_{D_C} |\rho_T|\, d\mu. \tag{9.7}$$

The integral $\int_X D(Vx, Px)\, d\mu(x)$ can now be estimated as follows.

$$\begin{aligned} \int_X D(Vx, Px)\, d\mu(x) &= \int_{\Psi} |\rho_V|\, d\mu \\ &\leq \int_{\Psi \setminus D_C} |\rho_V|\, d\mu + \int_{D_C \cap \Psi} |\rho_V|\, d\mu \\ \because Tx = Vx \text{ for } x \notin D_C \text{ and Eq. (9.7)} &\leq \int_{\Psi \setminus D_C} |\rho_T|\, d\mu + \int_{D_C} |\rho_T|\, d\mu. \end{aligned}$$

Finally, we consider the integral $\int_{\Psi\setminus D_C} |\rho_T|\,d\mu$ and partition its domain $\Psi \setminus D_C$ as $(J \cap (\Psi \setminus D_C)) \sqcup ((\Psi \setminus D_C) \setminus J)$, which yields

$$\begin{aligned}\int_{\Psi\setminus D_C} |\rho_T|\,d\mu &\le \int_J |\rho_T|\,d\mu + K\mu(\Psi)\\ &\le \int_J |\rho_T|\,d\mu + \frac{K(\beta-\gamma)}{\gamma}\mu(\operatorname{supp} T),\end{aligned}$$

where the last inequality follows from Lemma 9.12 with $\delta = 1 - \gamma/\beta$. Combining all the inequalities together, we get

$$\int_X D(Tx, Px)\,d\mu(x) \le 5\int_{D_C} |\rho_T|\,d\mu + \int_J |\rho_T|\,d\mu + \frac{K(\beta-\gamma)}{\gamma}\mu(\operatorname{supp} T). \qquad \blacksquare$$

Lemma 9.15 allows us to approximate, with a periodic transformation, a monotone T for which the Lebesgue measure of points jumping over any given $c \in X$ is roughly constant across orbits. To deal with the general case, we simply need to split the phase space X into countably many segments that are invariant under the flow and apply Lemma 9.15 on each of them separately. Small care needs to be taken to ensure that the values $(\beta-\gamma)/\gamma$, which appear in the formulation of Lemma 9.15, remain uniformly small across the partition of X. Details are presented in the following theorem. Let us recall that λ_c denotes the Lebesgue measure on the entire orbit of c, as discussed in Section 4.2 (the measure λ_c^C, which we have used throughout this chapter, corresponds to the Lebesgue measure restricted to the interval $c + [0, \operatorname{gap}_C(c))$).

Theorem 9.16. *Let $T \in [\mathbb{R} \curvearrowright X]_1$ be a monotone transformation that belongs to the kernel of the index map. For any $\epsilon > 0$, there exists a periodic transformation $P \in [\mathbb{R} \curvearrowright X]_1$ such that* $\operatorname{supp} P \subseteq \operatorname{supp} T$ *and* $\int_X D(Tx, Px)\,d\mu(x) < \epsilon$.

Proof. Without loss of generality, we assume $\epsilon \le 1$. Let $K_\epsilon \ge 1$ be such that for the set

$$J_\epsilon = \{x \in \operatorname{supp} T : |\rho_T(x)| \ge K_\epsilon\}$$

one has $\int_{J_\epsilon} |\rho_T|\,d\mu < \epsilon/18$. Pick a cross-section C with gaps so large that

$$2K_\epsilon^2/\operatorname{gap}_C(c) < \epsilon/15$$

for all $c \in C$, which ensures

$$K_\epsilon \cdot \mu(D_C \setminus J_\epsilon) \le \epsilon/15. \tag{9.8}$$

Note also that Eq. (9.8) holds for any cross-section $C' \subseteq C$, since $D_{C'} \subseteq D_C$ and $\operatorname{gap}_{C'}(c) \ge \operatorname{gap}_C(c)$ for all $c \in C'$.

For any positive real $\alpha > 0$, the positive real $\delta(\alpha) = \epsilon\alpha/(5 \cdot 3K_\epsilon)$ satisfies $\delta(\alpha) < \alpha$ and $2\delta(\alpha)/(\alpha - \delta(\alpha)) < \epsilon/3K_\epsilon$. We may therefore pick countably many positive reals $\alpha_n > 0$, $\delta_n > 0$, $n \geq 1$, such that $\mathbb{R}^{>0} = \bigcup_n (\alpha_n - \delta_n/2, \alpha_n + \delta_n/2)$ and

$$\left(\frac{2\delta_n}{\alpha_n - \delta_n}\right) < \frac{\epsilon}{3K_\epsilon} \quad \forall n \geq 1. \tag{9.9}$$

Define intervals $I_n = (\alpha_n - \delta_n/2, \alpha_n + \delta_n/2)$, $n \geq 1$.

Let $\zeta : \mathcal{C} \to \mathbb{R}^{\geq 0}$ be the map that measures the set of forward arcs over its argument:

$$\zeta(c) = \lambda_c\big(\{x \in \vec{X} : x < c \leq Tx\}\big).$$

Our assumption that T lies in the kernel of the index map implies that ζ also measures the set of backward arcs over its argument. Specifically, by Eq. (6.3) from Proposition 6.6, after discarding an invariant null set, we have for every $c \in \mathcal{C}$,

$$\lambda_c(\{x \in \operatorname{supp} T : x < c \leq Tx\}) = \lambda_c(\{x \in \operatorname{supp} T : Tx < c \leq x\}).$$

Set $\mathcal{C}_1 = \zeta^{-1}(I_1)$ and construct inductively $\mathcal{C}_n = \zeta^{-1}(I_n) \setminus \big[\bigcup_{k<n} \mathcal{C}_k\big]_{\mathcal{R}}$, where $\mathcal{R}$ denotes the orbit equivalence relation induced by the flow $\mathbb{R} \curvearrowright X$. The sets $\mathcal{C}_n$ are pairwise disjoint, and moreover, $(c_1, c_2) \notin \mathcal{R}$ for all $c_1 \in \mathcal{C}_{n_1}$, $c_2 \in \mathcal{C}_{n_2}$, $n_1 \neq n_2$. Let $\chi_n : \mathcal{C}_n \to \mathbb{N}$, $n \geq 1$, be the function defined by

$$\begin{aligned}\chi_n(c) = \min\Big\{m \in \mathbb{N} :& \\ &\lambda_c\big(\big\{x \in \vec{X} : x < c \leq Tx, D(x,c) \leq m, D(Tx,c) \leq m\big\}\big) > \zeta(c) - \delta_n/2 \\ &\text{and } \lambda_c\big(\big\{x \in \overleftarrow{X} : Tx < c \leq x, D(x,c) \leq m, D(Tx,c) \leq m\big\}\big) > \zeta(c) - \delta_n/2\Big\}.\end{aligned}$$

Set $\mathcal{C}'_{n,1} = \chi_n^{-1}(1)$ and define inductively $\mathcal{C}'_{n,m} = \chi^{-1}(m) \setminus \big[\bigcup_{k<m} \mathcal{C}'_{n,k}\big]_{\mathcal{R}}$. Let $X_{n,m}$ denote the saturated set $[\mathcal{C}'_{n,m}]_{\mathcal{R}}$. Finally, for all $m, n \geq 1$, let $\mathcal{C}_{n,m} \subseteq \mathcal{C}'_{n,m}$ be a subsection satisfying $\operatorname{gap}_{\mathcal{C}_{n,m}}(c) > m$ for all $c \in \mathcal{C}_{n,m}$. The sets $\mathcal{C}_{n,m}$ and $X_{n,m}$ satisfy the following conditions:

(1) $\mathcal{C}_{n,m}$ is a cross-section for the restriction of the flow onto $X_{n,m}$;

(2) the sets $X_{n,m}$, $m, n \geq 1$, are pairwise disjoint.

(3) $\zeta(c) \in I_n$ and, for all $c \in \mathcal{C}_{n,m}$, we have

$$\lambda_c^{\mathcal{C}_{n,m}}\big(\{x \in \vec{X} : x < \sigma_{\mathcal{C}_{n,m}}(c) \leq Tx\}\big) > \alpha_n - \delta_n \text{ and}$$
$$\lambda_c^{\mathcal{C}_{n,m}}\big(\{x \in \overleftarrow{X} : Tx < c \leq x\}\big) > \alpha_n - \delta_n.$$

Let $T_{n,m}$ denote the restriction of T onto $X_{n,m}$. Apply Lemma 9.15 to the transformation $T_{n,m}$, cross-section $\mathcal{C}_{n,m}$ with gaps of size at least K_ϵ, and $\beta = \alpha_n + \delta_n$, $\gamma = \alpha_n - \delta_n$. Let $P_{n,m}$ be the resulting periodic transformation on $X_{n,m}$. Set $P = \bigsqcup_{n,m} P_{n,m}$. We claim that P satisfies the conclusions of the theorem. Set $C' = \bigsqcup_{n,m} C_{n,m}$ and note that $C' \subseteq C$, whence $D_{C'} \subseteq D_C$. We can now split the integral $\int_X D(Tx, Px)\, d\mu(x)$ as:

$$\int_X D(Tx, Px)\, d\mu(x) = \sum_{n,m} \int_{X_{n,m}} D(T_{n,m}x, P_{n,m}x)\, d\mu(x).$$

Applying Lemma 9.15, we obtain the following chain of inequalities:

$$\begin{aligned}
\int_X D(Tx, Px)\, d\mu(x) &\leq 5 \sum_{n,m} \int_{D_{C_{n,m}}} |\rho_T|\, d\mu + \sum_{n,m} \int_{J_\epsilon \cap X_{n,m}} |\rho_T|\, d\mu \\
&\quad + \sum_{n,m} K_\epsilon \Big(\frac{2\delta_n}{\alpha_n - \delta_n}\Big) \mu(X_{n,m}) \\
\because \text{Eq. (9.9)} &\leq 5 \int_{D_C} |\rho_T|\, d\mu + \int_{J_\epsilon} |\rho_T|\, d\mu + (\epsilon/3)\mu(X) \\
&\leq 5 \int_{D_C \setminus J_\epsilon} |\rho_T|\, d\mu + 6 \int_{J_\epsilon} |\rho_T|\, d\mu + \epsilon/3 \\
\because \text{choice of } K_\epsilon &< 5K_\epsilon \mu(D_C \setminus J_\epsilon) + \epsilon/3 + \epsilon/3 \\
\because \text{Eq. (9.8)} &\leq \epsilon,
\end{aligned}$$

and the theorem follows. ■

Corollary 9.17. *Let $\mathbb{R} \curvearrowright X$ be a measure-preserving flow and $T \in [\mathbb{R} \curvearrowright X]_1$ be a dissipatively supported transformation. If $\mathcal{I}(T) = 0$, then $T \in \mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$.*

Proof. By Corollary 9.8, there is a monotone transformation U and a periodic transformation P such that $T = U \circ P$. Since $P \in \mathfrak{D}([\mathbb{R} \curvearrowright X]_1)$ by Corollary 3.16, it remains to show that U belongs to the derived subgroup. The latter follows from Theorem 9.16, since $\mathcal{I}(U) = \mathcal{I}(T) - \mathcal{I}(P) = 0$. ■

Chapter 10

Conclusions

Our objective in this last chapter is to draw several conclusions regarding the structure of the L^1 full groups of measure-preserving flows. The analysis conducted in Chapters 8 and 9 leads to the most technically challenging result of our work, which is the following theorem.

Theorem 10.1. *Let $\mathcal{F} : \mathbb{R} \curvearrowright X$ be a free measure-preserving flow on a standard probability space. The kernel of the index map coincides with the derived subgroup $\mathfrak{D}([\mathcal{F}]_1)$.*

Proof. The inclusion $\mathfrak{D}([\mathcal{F}]_1) \subseteq \ker \mathcal{I}$ is automatic since the image of $\mathcal{I}$ is abelian. For the other direction, pick a transformation $T \in \ker \mathcal{I}$ and consider its Hopf decomposition $X = C \sqcup D$ provided by Proposition 4.16. We have $T = T_C \circ T_D$, where $T_C \in [\mathcal{F}]_1$ is conservative and $T_D \in [\mathcal{F}]_1$ is dissipatively supported. According to Corollary 8.8, $\mathcal{I}(T_C) = 0$ and $T_C \in \mathfrak{D}([\mathcal{F}]_1)$, whence $\mathcal{I}(T_D) = \mathcal{I}(T) - \mathcal{I}(T_C) = 0$. Therefore, the dissipative part T_D satisfies the assumptions of Corollary 9.17, which yields $T_D \in \mathfrak{D}([\mathcal{F}]_1)$, and hence $T \in \mathfrak{D}([\mathcal{F}]_1)$ as desired. ∎

10.1 Topological ranks of L^1 full groups

Empowered with the result above and Theorem 5.19, we can estimate the topological ranks of L^1 full groups of flows. We recall the following well-known inequalities.

Proposition 10.2. *Let $\phi : G \to H$ be a surjective continuous homomorphism of Polish groups. The topological rank $\mathrm{rk}(G)$ satisfies*

$$\mathrm{rk}(H) \leq \mathrm{rk}(G) \leq \mathrm{rk}(H) + \mathrm{rk}(\ker \phi).$$

Proposition 10.3. *Let $\mathcal{F} : \mathbb{R} \curvearrowright X$ be a free measure-preserving flow on a standard probability space (X, μ). The topological rank $\mathrm{rk}([\mathcal{F}]_1)$ is finite if and only if the flow has finitely many ergodic components. Moreover, if $\mathcal{F}$ has exactly n ergodic components, then*

$$n + 1 \leq \mathrm{rk}([\mathcal{F}]_1) \leq n + 3.$$

Proof. Let $\mathcal{E}$ be the space of probability invariant ergodic measures of the flow, and let p be the probability measure on $\mathcal{E}$ such that $\mu = \int_{\mathcal{E}} \nu \, dp(\nu)$ (see Appendix E.1). Proposition 6.6 shows that the index map $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E}, p)$ is continuous and surjective. An application of Proposition 10.2 yields

$$\mathrm{rk}(\mathrm{L}^1(\mathcal{E},p)) \leq \mathrm{rk}([\mathcal{F}]_1) \leq \mathrm{rk}(\mathrm{L}^1(\mathcal{E},p)) + \mathrm{rk}(\ker \mathcal{I}) = \mathrm{rk}(\mathrm{L}^1(\mathcal{E},p)) + 2, \quad (10.1)$$

where the last equality is based on Theorem 10.1 and Theorem 5.19. Since $\mathrm{L}^1(\mathcal{E},p)$ is a Banach space, its topological rank is finite if and only if its dimension is finite, which is equivalent to $(\mathcal{E},p)$ being purely atomic with finitely many atoms. We have shown that $\mathrm{rk}([\mathcal{F}]_1)$ is finite if and only if the flow has only finitely many ergodic components. The moreover part of the proposition follows from the inequality (10.1) and the observation that $\mathrm{rk}(\mathrm{L}^1(\mathcal{E},p)) = \dim(\mathrm{L}^1(\mathcal{E},p)) + 1$. ∎

As already mentioned in the introduction, we conjecture that the topological rank completely remembers the number of ergodic components.

Conjecture 10.4. *Let $\mathcal{F}$ be a free measure-preserving flow. If it has exactly n ergodic components, then* $\mathrm{rk}([\mathcal{F}]_1) = n + 1$.

Provided the conjecture holds, we have a priori no way of distinguishing L^1 full groups of ergodic flows as topological groups. For $\mathbb{Z}$-actions, it is a consequence of Belinskaja's theorem that there are many L^1 full groups. The following two sections explore analogues of her result for flows, demonstrating the existence of numerous L^1 full groups associated with free ergodic flows. While we currently lack a concrete method to distinguish these groups, their geometric properties—discussed in the final section—may provide valuable insights in this direction.

10.2 Katznelson's conjugation theorem

R. M. Belinskaja [8] showed that if measure-preserving transformations $T, U \in \mathrm{Aut}(X,\mu)$ generate the same orbit equivalence relation, i.e., $\mathcal{R}_T = \mathcal{R}_U$, and $U \in [T]_1$, then T and U are conjugated. Y. Katznelson found a different argument and isolated a sufficient condition for the conjugacy of measure-preserving transformations (see [10, Theorem A.1]). In the following, for $T \in \mathrm{Aut}(X,\mu)$, $x \in X$, and $A \subseteq \mathbb{Z}$, we let $T^A x$ denote the set $\{T^k x : k \in A\}$.

Theorem 10.5 (Katznelson). *Suppose $T, U \in \mathrm{Aut}(X,\mu)$ are measure-preserving transformations that generate the same orbit equivalence relation, $\mathcal{R}_T = \mathcal{R}_U$. If the symmetric difference $T^{\mathbb{N}}x \vartriangle U^{\mathbb{N}}x$ is finite for almost all x, then T and U are conjugated by an element from the full group $[T] = [U]$.*

The analog of this result for free measure-preserving flows will be proved shortly in Theorem 10.9. But first, we discuss an important application of Theorem 10.5. Consider a free measure-preserving flow $\mathcal{F} : \mathbb{R} \curvearrowright X$. Given a dissipatively supported transformation $T \in [\mathcal{F}]$ (in the sense of Definition 9.1), Proposition C.4 implies that almost every non-trivial T-orbit $[x]_{\mathcal{R}_T}$ is a discrete subset of $[x]_{\mathcal{R}}$ unbounded both from below and from above. The order induced on $[x]_{\mathcal{R}_T}$ by the flow may disagree with the T-order of points. One may therefore define the **$\mathcal{F}$-reordering** of T to be the first return transformation $\tilde{T}$ induced by the ordering of the flow on the orbits of T:

$$\tilde{T}x = x + \min\{r > 0 : x + r \in [x]_{\mathcal{R}_T}\} \qquad \text{for } x \in \operatorname{supp} T.$$

Note that T and $\tilde{T}$ generate the same orbit equivalence relation, $\mathcal{R}_T = \mathcal{R}_{\tilde{T}}$.

If T belongs to the L^1 full group of the flow, either $T^{\mathbb{N}}x \bigtriangleup \tilde{T}^{\mathbb{N}}x$ or $T^{\mathbb{N}}x \bigtriangleup \tilde{T}^{-\mathbb{N}}x$ is finite, depending on whether $\lim_n \rho(x, T^n x) = +\infty$ or $\lim_n \rho(x, T^n x) = -\infty$ (cf. Corollary 9.4). Which symmetric difference is finite may depend on the point $x \in X$, and Theorem 10.5 can be used to show that T and its reordering $\tilde{T}$ are flip-conjugate.

Definition 10.6. Let (X_1, μ_1) and (X_2, μ_2) be standard probability spaces, and let $T_i \in \mathrm{Aut}(X_i, \mu_i)$, $i = 1, 2$. Measure-preserving transformations T_1 and T_2 are **flip-conjugate** if there exist an isomorphism of measure spaces $S : X_1 \to X_2$ and measurable partitions $X_1 = X_1^- \sqcup X_1^+$, $X_2 = X_2^- \sqcup X_2^+$ such that

(1) $S(X_1^-) = X_2^-$ and $S(X_1^+) = X_2^+$;

(2) X_1^-, X_1^+ are T_1-invariant, and X_2^-, X_2^+ are T_2-invariant;

(3) $ST_1 \restriction_{X_1^+} S^{-1} = T_2 \restriction_{X_2^+}$ and $ST_1 \restriction_{X_1^-} S^{-1} = T_2^{-1} \restriction_{X_2^-}$.

Note that when one of the T_i's is ergodic, our definition of flip-conjugacy coincides with the standard one, which requires X_i^- or X_i^+ to have full measure.

Proposition 10.7. *Any dissipatively supported $T \in [\mathcal{F}]_1$ and its $\mathcal{F}$-reordering $\tilde{T}$ are flip-conjugated by an element from the full group $[T] = [\tilde{T}]$.*

Proof. Consider the decomposition $\operatorname{supp} T = \overleftarrow{X} \sqcup \overrightarrow{X}$ into the positive and negative orbits as in Definition 9.5. In particular, $T^{\mathbb{N}}x \triangle \tilde{T}^{\mathbb{N}}x$ and $T^{\mathbb{N}}x \triangle \tilde{T}^{-\mathbb{N}}x$ are finite for $x \in \overrightarrow{X}$ and $x \in \overleftarrow{X}$, respectively. Theorem 10.5 implies that there exist automorphisms $S_1 \in [T \restriction_{\overrightarrow{X}}]$ and $S_2 \in [T \restriction_{\overleftarrow{X}}]$ such that $S_1 T \restriction_{\overrightarrow{X}} S_1^{-1} = \tilde{T} \restriction_{\overrightarrow{X}}$ and $S_2 T \restriction_{\overleftarrow{X}} S_2^{-1} = \tilde{T}^{-1} \restriction_{\overleftarrow{X}}$. The transformation S given by

$$Sx = \begin{cases} S_1 x & \text{if } x \in \overrightarrow{X}, \\ S_2 x & \text{if } x \in \overleftarrow{X}, \\ x & \text{otherwise} \end{cases}$$

belongs to the full group $[T]$ and witnesses the flip-conjugacy of T and $\tilde{T}$. ∎

The transformation conjugating T and U in Theorem 10.5 can be written fairly explicitly. This is done in terms of the function δ defined as follows. Suppose (Ω, λ) is a (possibly infinite) measure space, and let $A, B \subseteq \Omega$ be measurable sets such that $\lambda(A \bigtriangleup B) < +\infty$. We set $\delta(A, B) = \lambda(A \setminus B) - \lambda(B \setminus A)$. This function satisfies a few properties which the reader can easily verify.

Proposition 10.8. *Suppose* (Ω, λ) *is a measure space. For all* $A, B, C, a \subseteq \Omega$ *such that* $\lambda(A \triangle B), \lambda(B \triangle C), \lambda(A \triangle C), \lambda(a) < +\infty$*, the following holds:*

(1) $\delta(A, C) = \delta(A, B) + \delta(B, C)$*;*

(2) $\delta(A, A) = 0$ *and* $\delta(A, B) = -\delta(B, A)$*;*

(3) $\delta(A \triangle a, B) = \delta(A, B) + (\lambda(a) - 2\lambda(a \cap A))$.

Any orbit of a measure-preserving transformation can be endowed with a counting measure. Given T and U as in the statement of Theorem 10.5, set $\tau(x) = \delta(U^{\mathbb{N}}x, T^{\mathbb{N}}x)$ and define $Sx = U^{\tau(x)}x$. One can verify that $S \in [U] = [T]$ and $STS^{-1} = U$ (further details can be found in [10, Theorem A.1]).

Let now $\mathcal{F}_1$ and $\mathcal{F}_2$ be measure-preserving flows on a standard probability space (X, μ); we denote the actions of $r \in \mathbb{R}$ upon $x \in X$ by $x +_1 r$ and $x +_2 r$, respectively. Suppose that their full groups coincide, $[\mathcal{F}_1] = [\mathcal{F}_2]$, and so the flows share the same orbits, $\mathcal{R}_{\mathcal{F}_1} = \mathcal{R}_{\mathcal{F}_2}$. For $x \in X$, let $s_i(x) = x +_i [0, \infty)$, $i = 1, 2$, denote the "right half-orbit" of x. A natural analog of the condition $|T^{\mathbb{N}}x \triangle U^{\mathbb{N}}x| < \infty$ from Theorem 10.5 would be to require finiteness of the Lebesgue measure of $s_1(x) \triangle s_2(x)$ for all $x \in X$. This condition alone, however, is not sufficient for the conjugacy of $\mathcal{F}_1$ and $\mathcal{F}_2$.

Each flow induces a copy of the Lebesgue measure onto orbits via

$$\lambda_{i,x}(A) = \lambda(\{r \in \mathbb{R} : x +_i r \in A\}).$$

Since we assume $[\mathcal{F}_1] = [\mathcal{F}_2]$, and so $\mathcal{F}_2 \subseteq [\mathcal{F}_1]$, $\lambda_{1,x}$ is a translation invariant measure relative to the action of $\mathcal{F}_2$, and therefore must differ from $\lambda_{2,x}$ by a constant: there is an orbit invariant measurable function $c : X \to \mathbb{R}^{>0}$ such that $\lambda_{2,x} = c(x)\lambda_{1,x}$. Any element in $[\mathcal{F}_1] = [\mathcal{F}_2]$ preserves $\lambda_{i,x}$, $i = 1, 2$, and therefore cannot conjugate $\mathcal{F}_1$ into $\mathcal{F}_2$ unless $c(x)$ is constantly equal to 1.

When the flows are ergodic, $c(x) = c$ is a constant, and one may renormalize the flows without changing the full groups. Let $\mathcal{F}_2'$ be the rescaling of $\mathcal{F}_2$ given by $x +_2' r = x +_2 cr$. It is straightforward to check that $\lambda'_{2,x}(A) = c^{-1}\lambda_{2,x}(A) = \lambda_{1,x}(A)$, and the flows $\mathcal{F}_1$ and $\mathcal{F}_2'$ induce the same measure onto orbits.

After this renormalization, the finiteness of the measure $s_1(x) \triangle s_2(x)$ for all $x \in X$ is indeed sufficient to establish the conjugacy of the flows.

Theorem 10.9. *Let* $\mathcal{F}_i$*,* $i = 1, 2$*, be free measure-preserving flows that share the same orbits,* $\mathcal{R}_{\mathcal{F}_1} = \mathcal{R}_{\mathcal{F}_2}$*, and induce the same measures* $(\lambda_x)_{x \in X}$ *onto orbits. If* $\lambda_x(s_1(x) \triangle s_2(x)) < +\infty$*,* $x \in X$*, then the flows are conjugate by a measure-preserving transformation* $S \in [\mathcal{F}_1]$.

Proof. Let $n : X \times \mathbb{R} \to \mathbb{R}$ be the $\mathcal{F}_1, \mathcal{F}_2$-cocycle defined by $x +_2 r = x +_1 n(x, r)$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ induce the same measure on the orbits, $n(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measure-preserving automorphism:

$$\begin{aligned}\lambda(n(x,A)) &= \lambda_{1,x}(\{x +_1 n(x,r) : r \in A\})\\ &= \lambda_{2,x}(\{x +_2 r : r \in A\}) = \lambda(A).\end{aligned}$$

For $x \in X$ and $r \in \mathbb{R} \cup \{+\infty\}$ let

$$s_{i,r}(x) = \begin{cases} x +_i [0,r) & \text{if } r \geq 0,\\ x +_i [r,0) & \text{if } r < 0.\end{cases}$$

In particular, $s_i(x) = s_{i,+\infty}(x)$. Note that

$$\begin{aligned} s_1(x +_2 r) &= s_1(x) \vartriangle s_{1,n(x,r)}(x),\\ s_2(x +_2 r) &= s_2(x) \vartriangle s_{2,r}(x).\end{aligned} \tag{10.2}$$

Also, considering the cases $r < 0$ and $r \geq 0$ separately, one can easily verify that for all $r \in \mathbb{R}$ and $i = 1, 2$

$$\lambda_{i,x}(s_{i,r}(x)) - 2\lambda_{i,x}(s_2(x) \cap s_{2,r}(x)) = -r.$$

and, in particular,

$$\begin{aligned}\lambda_{1,x}(s_{1,n(x,r)}(x)) - 2\lambda_{1,x}(s_1(x) \cap s_{1,n(x,r)}(x)) &= -n(x,r),\\ \lambda_{2,x}(s_{2,r}(x)) - 2\lambda_{2,x}(s_2(x) \cap s_{2,r}(x)) &= -r.\end{aligned} \tag{10.3}$$

Put $\tau(x) = \delta(s_1(x), s_2(x))$, then

$$\begin{aligned} \tau(x +_2 r) &= \delta(s_1(x +_2 r), s_2(x +_2 r))\\ \because \text{Eq. (10.2)} &= \delta(s_1(x) \vartriangle s_{1,n(x,r)}(x), s_2(x +_2 r))\\ \because \text{Prop. 10.8} &= \delta(s_1(x), s_2(x +_2 r)) +\\ &\qquad \lambda_{1,x}(s_{1,n(x,r)}(x)) - 2\lambda_{1,x}(s_1(x) \cap s_{1,n(x,r)}(x))\\ \because \text{Eq. (10.3)} &= \delta(s_1(x), s_2(x +_2 r)) - n(x,r)\\ \because \text{Prop. 10.8} &= -\delta(s_2(x +_2 r), s_1(x)) - n(x,r) = \delta(s_1(x), s_2(x +_2 r)) -\\ &\qquad (\lambda_{2,x}(s_{2,r}(x)) - 2\lambda_{2,x}(s_2(x) \cap s_{2,r}(x))) - n(x,r)\\ \because \text{Eq. (10.3)} &= \delta(s_1(x), s_2(x)) - n(x,r) + r.\end{aligned} \tag{10.4}$$

The required transformation $S : X \to X$ is given by $Sx = x +_1 \tau(x)$.

$$\begin{aligned} S(x +_2 r) &= (x +_2 r) +_1 \tau(x +_2 r) = (x +_1 n(x,r)) +_1 \tau(x +_2 r)\\ \because \text{Eq. (10.4)} &= x +_1 (n(x,r) + \tau(x) - n(x,r) + r) = Sx +_1 r.\end{aligned}$$

Thus S conjugates $\mathcal{F}_1$ and $\mathcal{F}_2$. It therefore remains to check that S is a measure-preserving bijection. First, note that Sx satisfies $\delta(s_1(Sx), s_2(x)) = 0$. Indeed, $s_1(Sx) = s_1(x) \vartriangle s_{1,\tau(x)}(x)$ (by the analog of Eq. (10.2)), and therefore

$$\delta(s_1(Sx), s_2(x)) = \tau(x) - \tau(x) = 0 \tag{10.5}$$

by Proposition 10.8.

To show injectivity, suppose that $Sx = Sy$. In view of Eq. (10.5) and Proposition 10.8,

$$\delta(s_2(x), s_2(y)) = \delta(s_2(x), s_1(Sx)) + \delta(s_1(Sy), s_2(y)) = 0.$$

However, if $y = x +_2 r$, then $s_2(y) = s_2(x) \vartriangle s_{2,r}(x)$ and so $\delta(s_2(x), s_2(y)) = r$. One concludes that $r = 0$ and $x = y$. We have already established that $S(x +_2 r) = Sx +_1 r$, which shows that the range of S is orbit invariant, yielding surjectivity.

Finally, to show that S is measure-preserving, it suffices to check that S preserves the Lebesgue measure $\lambda_{1,x} = \lambda_{2,x}$ on all the orbits. To this end, let $n' : X \times \mathbb{R} \to \mathbb{R}$ be the $\mathcal{F}_1$-cocycle (i.e., $x +_1 r = x +_2 n'(x, r)$). For all $r' \in \mathbb{R}$, one has

$$\begin{aligned}
\lambda_{1,x}(Ss_{1,r'}(x)) &= \lambda_{1,x}(\{y +_1 \tau(y) : y \in s_{1,r'}(x)\}) \\
&= \lambda_{1,x}(\{(x +_1 r) +_1 \tau(x +_1 r) : 0 \le r < r'\}) \\
&= \lambda(\{r + (\tau(x) - r + n'(x, r)) : 0 \le r < r'\}) \\
&= \lambda(\{n'(x, r) : 0 \le r < r'\}) = \lambda(n'(x, [0, r'))) = r'.
\end{aligned}$$

Hence, $S \in \mathrm{Aut}(X, \mu)$ is the required conjugation between $\mathcal{F}_1$ and $\mathcal{F}_2$. ■

In the $\mathbb{Z}$ case, the above result is the key to Belinskaja's flip-conjugacy result for L^1 orbit equivalence. Unfortunately, we do not know if it can be useful for proving an analogous result for flows. In the next section, we nevertheless obtain a weaker result that shows there are many L^1 full groups. We leave the following question open.

Question 10.10. *Given two ergodic flows with equal* L^1 *full groups, does there exist a rescaling under which they satisfy the hypothesis of the above theorem?*

10.3 L^1 orbit equivalence implies flip Kakutani equivalence

A measure-preserving action of a compactly generated locally compact Polish group can always be twisted by a continuous automorphism of the group without affecting the L^1 full group.

In the case of $\mathbb{Z}$-actions, this takes a particularly simple form since the only non-trivial automorphism of $\mathbb{Z}$ is given by $n \mapsto -n$. It follows from the results of R. M. Belinskaja [8] that this is, up to conjugacy, the only way to get an L^1 orbit equivalence for ergodic $\mathbb{Z}$-actions [40, Theorem 4.2]: if T_1, T_2 are two ergodic measure-preserving transformations that are L^1 orbit equivalent, then they are flip-conjugate: T_1 is conjugate to either T_2 or T_2^{-1}.

As mentioned before, we do not know whether a variant of such rigidity holds when we replace $\mathbb{Z}$ by $\mathbb{R}$ (see Question 10.17 below), but, as shown in Theorem 10.15, L^1 orbit equivalent free measure-preserving flows must at least be flip Kakutani equivalent. In particular, there are uncountably many L^1 full groups of free ergodic flows up to abstract group isomorphism.

Let us first define the notion of (flip) Kakutani equivalence of flows. For the main results about this concept, the reader may consult [28, 29], where it is called **monotone equivalence** of flows. Given a measure-preserving automorphism $T \in \mathrm{Aut}(Z, \nu)$ and a positive integrable function $f \in L^1(Z, \nu)$, one can define the so-called **suspension flow** or **flow under a function** on the space

$$X = \{(z, t) : z \in Z,\ 0 \leq t < f(z)\}.$$

For $r \geq 0$, the action $(z, t) + r$ is given by

$$(z, t) + r = \Big(T^k z, t + r - \sum_{i=0}^{k-1} f(T^i z)\Big),$$

where $k \geq 0$ is defined uniquely by the condition $\sum_{i=0}^{k-1} f(T^i z) \leq t + r < \sum_{i=0}^{k} f(T^i z)$; similarly, for $r \leq 0$ the action is

$$(z, t) + r = \Big(T^{-k} z, t + r + \sum_{i=1}^{k} f(T^{-i} z)\Big),$$

where $k \geq 0$ satisfies $0 \leq t + r + \sum_{i=1}^{k} f(T^{-i} z) < f(T^{-k} z)$. Such a flow preserves the restriction onto X of the product measure $\nu \times \lambda$. The space (X, μ), where

$$\mu = \frac{\nu \times \lambda}{\int_Z f \, d\nu} \restriction_X,$$

is a standard probability space. The automorphism T in the suspension flow construction is called the **base automorphism**.

Definition 10.11. Two flows are **flip Kakutani equivalent** if they are isomorphic to suspension flows over flip-conjugate base automorphisms.

It is important to note that the construction of suspension flows can be reversed through the use of cross-sections[1]. Given a free flow on (X, μ) and a cocompact U-

[1] In full generality, the definition of a cross-section should actually be relaxed, replacing lacunarity with discreteness in each orbit, and only requiring the gap function of the cross-section to be integrable.

lacunary cross-section $C \subseteq X$ for a precompact neighborhood of the identity $U \subseteq G$, there is a unique finite measure ν on C such that the map $U \times C \to C + U \subseteq X$ taking (t, c) to $c + t$ is measure-preserving (see [37, Prop. 4.3] for the general construction). The first-return map $\sigma_C : C \to C$ is measure-preserving, and our initial flow can be seen as the flow built under the gap function gap_C with the base transformation σ_C.

We require the following key result, established by D. Rudolph [54]. In light of the preceding discussion, it can be restated as follows: every free measure-preserving flow is conjugate to a suspension flow with a two-valued function.

Theorem 10.12 (Rudolph). *Let $t_0 \in \mathbb{R}^{>0} \setminus \mathbb{Q}$ be a positive irrational number. Any free measure-preserving flow on a standard probability space admits a cross-section whose gap function takes only the values 1 and t_0 almost surely.*

Remark 10.13. The second-named author has obtained a generalization of this result in the purely Borel context; see [57].

Theorem 10.14. *Let $\mathcal{F}, \mathcal{F}'$ be free measure-preserving flows on (X, μ) that share the same orbits, namely $\mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{F}'}$. If $\mathcal{F}' \leq [\mathcal{F}]_1$, then $\mathcal{F}$ and $\mathcal{F}'$ are flip Kakutani equivalent.*

Proof. We denote the flow $\mathcal{F}$ using our usual notation, $(x, t) \mapsto x + t$. As explained right after Definition 10.11, it suffices to find cross-sections for $\mathcal{F}$ and $\mathcal{F}'$ such that the corresponding first return automorphisms are flip-conjugate.

Fix some irrational $t_0 > 1$, and let $C \subseteq X$ be a Borel cross-section for $\mathcal{F}$ such that, outside a Borel $\mathcal{F}$-invariant null set, we have $\mathrm{gap}_C(c) \in \{1, t_0\}$ for all $c \in C$, as provided by Theorem 10.12. Define the automorphism $T : X \to X$ by

$$Tx = \begin{cases} \sigma_C(c) + \alpha & \text{if } x = c + \alpha \text{ for some } c \in C,\ \alpha \in [0, 1], \\ x & \text{otherwise.} \end{cases}$$

The transformation T is obtained by gluing together the identity map, $x \mapsto x + 1$, and $x \mapsto x + t_0$. Since all these belong to $[\mathcal{F}]_1$, which is finitely full, we have $T \in [\mathcal{F}]_1$ as well. Note that T is dissipatively supported and is therefore flip-conjugate to its $\mathcal{F}'$-reordering $\tilde{T}$ by Proposition 10.7. In other words, there is a T-invariant Borel set $Z \subseteq X$ of full measure, $\mu(Z) = 1$, and a T-invariant Borel partition $Z = Z^+ \sqcup Z^-$ such that $T \restriction_{Z^+}$ is conjugate to $\tilde{T} \restriction_{Z^+}$ and $T \restriction_{Z^-}$ is conjugate to $\tilde{T}^{-1} \restriction_{Z^-}$.

Let ν be the measure on C given for a Borel $A \subseteq C$ by $\nu(A) = \mu(A + [0, 1))$. The measure $\mu \restriction_{C+[0,1)}$ is naturally isomorphic to $(\nu \times \lambda) \restriction_{C+[0,1)}$, where λ is the Lebesgue measure on $[0, 1]$, and we therefore have

$$\forall^{\nu \times \lambda} (c, \lambda) \in C \times [0, 1) \quad c + \alpha \in Z.$$

By Fubini's theorem, this is equivalent to

$$\forall^{\lambda} \alpha \in [0, 1)\ \forall^{\nu} c \in C \quad (c + \alpha \in Z).$$

Therefore, there exists some $\alpha_0 \in [0,1)$ such that $\nu(\{c \in C : c + \alpha_0 \in Z\}) = 1$. Note that $T\restriction_{C+\alpha_0}$ is the first return map on $C+\alpha_0$ in the order of the flow $\mathcal{F}$, whereas $\tilde{T}\restriction_{C+\alpha_0}$ is the first return map in the order induced on the orbits by $\mathcal{F}'$. Since $T\restriction_{C+\alpha_0}$ and $\tilde{T}\restriction_{C+\alpha_0}$ are flip-conjugate, the flows are flip Kakutani equivalent. ∎

Theorem 10.14 has the following straightforward consequences.

Corollary 10.15. *If two free ergodic measure-preserving flows are* L^1 *orbit equivalent, then they are also flip Kakutani equivalent.*

Proof. This now follows from the definition of L^1 orbit equivalence, see Definition 4.19 and the paragraph thereafter. ∎

Corollary 10.16. *If two free ergodic measure-preserving flows have abstractly isomorphic* L^1 *full groups, then they are also flip Kakutani equivalent.*

Proof. We have seen in Proposition 4.21 that the isomorphism of L^1 full groups of ergodic flows implies L^1 orbit equivalence, so the result follows from the previous corollary. ∎

Kakutani equivalence is a highly non-trivial equivalence relation (see [49] or [21, 36]). It seems likely, however, that L^1 full groups of flows contain even more information about the action. The only continuous automorphisms of $\mathbb{R}$ are multiplications by nonzero scalars, and we ask whether the isomorphism of L^1 full groups necessarily recovers the action up to such an automorphism.

Question 10.17. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be free ergodic measure-preserving flows with isomorphic* L^1 *full groups. Is it true that there exists $\alpha \in \mathbb{R}^*$ such that $\mathcal{F}_1$ and $\mathcal{F}_2 \circ m_\alpha$ are isomorphic, where m_α denotes the multiplication by α?*

Note that a positive answer to Question 10.10 would imply a positive answer to the above question.

10.4 Maximality of the L^1 norm and geometry

In this last section, we show that the L^1 norm is maximal on L^1 full groups of flows. In particular, it defines their quasi-isometry type. Exploring this quasi-isometry type further might lead to topological group invariants capable of distinguishing some ergodic flows.

Theorem 10.18. *Let $\mathcal{F}$ be a free measure-preserving flow. The* L^1 *norm on* $[\mathcal{F}]_1$ *is maximal.*

Proof. We have already shown that the L^1 norm on the derived L^1 full group is maximal (see Theorem 5.5). Denote by $(\mathcal{E}, p)$ the space of $\mathcal{F}$-invariant ergodic probability measures, where p is the probability measure arising from the disintegration of μ, which we write as $x \mapsto \nu_x$ (see Section E.1). The derived L^1 full group is equal to the kernel of the surjective index map $\mathcal{I} : [\mathcal{F}]_1 \to \mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$ and the quotient norm on $[\mathcal{F}]_1/\ker I$ is equal to the L^1 norm on $\mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$ by Proposition 6.7. The latter norm is maximal, as is the case for any Banach space norm.

Given a function $f \in \mathrm{L}^1(\mathcal{E}, p, \mathbb{R})$, let $U_f \in [\mathcal{F}]_1$ be given by $U_f(x) = x + f(\nu_x)$. The cocycle $\rho_{U_f}(x) = f(\nu_x)$ is constant on each ergodic component and $\|U_f\|_1 = \|f\|_1$. Furthermore, $\mathcal{I}(U_f) = f$. We show that $\|\cdot\|$ is both large-scale geodesic and coarsely proper (see Appendix A.2 and Proposition A.10, in particular).

Any $T \in [\mathcal{F}]_1$ can be written as $T = (TU^{-1}_{\mathcal{I}(T)})U_{\mathcal{I}(T)}$, where the transformation $TU^{-1}_{\mathcal{I}(T)} \in \ker \mathcal{I} = \mathfrak{D}([\mathcal{F}]_1)$, and $\|U_{\mathcal{I}(T)}\|_1 \leq \|T\|_1$. In particular, we have $\|TU^{-1}_{\mathcal{I}(T)}\|_1 \leq 2\|T\|_1$.

Since the L^1 norm is maximal on $\mathfrak{D}([\mathcal{F}]_1)$, it is large-scale geodesic. In fact, Proposition 3.25 establishes that it is large-scale geodesic with constant $K = 2$. We may therefore express $TU^{-1}_{\mathcal{I}(T)}$ as a product $V_1 \cdots V_n$ of elements $V_i \in \mathfrak{D}([\mathcal{F}]_1)$, where each V_i has norm at most K and

$$\sum_{i=1}^{n} \|V_i\|_1 \leq K\|TU^{-1}_{\mathcal{I}(T)}\|_1 \leq 2K\|T\|_1.$$

The transformation $U_{\mathcal{I}(T)}$ can, for any $m \geq 1$, also be expressed as a product

$$U_{\mathcal{I}(T)} = U_{\mathcal{I}(T)/m} \cdots U_{\mathcal{I}(T)/m} = U^m_{\mathcal{I}(T)/m}.$$

By taking m sufficiently large, we can ensure that $\|U_{\mathcal{I}(T)/m}\|_1 = \|\mathcal{I}(T)/m\|_1 \leq K$. Therefore, $T = (V_1 \cdots V_n)(U_{\mathcal{I}(T)/m} \cdots U_{\mathcal{I}(T)/m})$, and

$$\sum_{i=1}^{n} \|V_i\|_1 + \sum_{j=1}^{m} \|U_{\mathcal{I}(T)/m}\|_1 \leq 2K\|T\|_1 + \|U_{\mathcal{I}(T)}\|_1 \leq 3K\|T\|_1.$$

We conclude that the norm $\|\cdot\|$ on $[\mathcal{F}]_1$ is large-scale geodesic with $K' = 3K = 6$.

It remains to prove coarse properness. Let $\epsilon > 0$ and $R > 0$ be positive reals. By Theorem 5.5, there exists $n \in \mathbb{N}$ so large that every element in the derived L^1 full group of norm at most $2R$ is a product of n elements of norm at most ϵ. Let N be any integer greater than R/ϵ. We argue that every element of $[\mathcal{F}]_1$ of norm at most R is a product of $2n + N$ elements of norm at most ϵ.

Indeed, if $T = (TU^{-1}_{\mathcal{I}(T)})U_{\mathcal{I}(T)}$ has norm at most R, then

$$\|TU^{-1}_{\mathcal{I}(T)}\|_1 \leq 2\,\|T\|_1 \leq 2R,$$

and $TU_{\mathcal{I}(T)}^{-1}$ can therefore by written as a product of n elements of $\mathfrak{D}([F]_1)$ each of norm $\leq \epsilon$. Moreover, $U_{\mathcal{I}(T)} = U_{\mathcal{I}(T)/N}^N$ and $\|U_{\mathcal{I}(T)/N}\|_1 \leq \epsilon$ by the choice of N. The conclusion follows. ■

Remark 10.19. While the proposition above states that L^1 full groups of flows are quite large, one can use Proposition 6.8 to show that they satisfy the *Haagerup property*. In other words, such groups admit a coarsely proper affine action on a Hilbert space (namely, the affine Hilbert space $\chi_{\mathcal{R}^{\geq 0}} + \mathrm{L}^2(\mathcal{R}, M)$).

Corollary 10.16 along with [49, Sec. 12] implies that there are uncountably many L^1 full groups of ergodic free flows up to topological group isomorphism. It would be interesting if their geometry allowed us to distinguish these groups. However, we do not even know the answer to the following question.

Question 10.20. *Are there two free ergodic measure-preserving flows with non-quasi-isometric* L^1 *full groups?*

Appendix A

Normed groups

We chose to present our work in the framework of groups equipped with compatible norms rather than metrics. These two frameworks are equivalent, but the former has some stylistic advantages, in our opinion. In Appendix A, we remind the reader the concept of a norm on a group (Section A.1) and state C. Rosendal's results on maximal norms (Section A.2).

A.1 Norms on groups

Definition A.1. A **norm** on a group G is a map $\|\cdot\| : G \longrightarrow \mathbb{R}^{\geq 0}$ such that for all $g, h \in G$

(1) $\|g\| = 0$ if and only if $g = e$;

(2) $\|g\| = \|g^{-1}\|$;

(3) $\|gh\| \leq \|g\| + \|h\|$.

If G is moreover a topological group, a norm $\|\cdot\|$ on G is called **compatible** if the balls $\{g \in G : \|g\| < r\}$, $r > 0$, form a basis of neighborhoods of the identity. When G is a Polish group equipped with a compatible norm $\|\cdot\|$, we refer to the pair $(G, \|\cdot\|)$ as a **Polish normed group**.

There is a correspondence between (compatible) left-invariant metrics on a group and (compatible) norms on it. Indeed, given a left-invariant metric d on G, the function $\|g\| = d(e, g)$ is a norm. Conversely, from a norm $\|\cdot\|$ one can recover the left-invariant metric d via $d(g, h) = \|g^{-1}h\|$. Analogously, there is a correspondence between norms and right-invariant metrics given by $d(g, h) = \|hg^{-1}\|$.

The language of group norms thus contains the same information as the formalism of left-invariant (or right-invariant) metrics, but it has the stylistic advantage of removing the need of making a choice between the invariant side, when such a choice is immaterial.

Remark A.2. Note, however, that there are metrics that are neither left- nor right-invariant, which nonetheless induce a group norm via the same formula $\|g\| = d(g, e)$. Consider for example a Polish group G with a compatible left-invariant metric d' on it. If G is not a CLI group, the metric d' is not complete, but the metric

$$d(f, g) = \frac{d'(f, g) + d'(f^{-1}, g^{-1})}{2}$$

is complete. Since $d(g, e) = d'(g, e)$, we see that d induces the same norm $\|\cdot\|$ as does the left-invariant metric d'.

There is a canonical way to push a norm onto a factor group.

Proposition A.3 (see [18, Thm. 2.2.10]). *Let $(G, \|\cdot\|)$ be a Polish normed group, and let $H \trianglelefteq G$ be a closed normal subgroup of G. The function*

$$\|gH\|^{G/H} = \inf\{\|gh\| : h \in H\}$$

is a norm on G/H which is compatible with the quotient topology. In particular, $(G/H, \|\cdot\|^{G/H})$ is a Polish normed group.

Definition A.4. A compatible norm $\|\cdot\|$ on a locally compact Polish group G is **proper** if all balls $\{g \in G : \|g\| \leq r\}$ are compact.

R. A. Struble [58] showed that all locally compact Polish groups admit a compatible proper norm.

A.2 Maximal norms

As we noted in Lemma 2.13, quasi-isometric norms yield the same L^1 full groups. C. Rosendal identified the class of Polish groups that admit maximal norms, which are unique up to quasi-isometry. In this section, we state some results from C. Rosendal's treatise [52], which are relevant to our work. For the reader's convenience, we formulate the following definitions and propositions in the language of group norms as opposed to left-invariant metrics or écarts, as in the original reference.

Definition A.5 ([52, Def. 2.68]). A compatible norm $\|\cdot\|$ on a Polish group G is said to be **maximal** if for any compatible norm $\|\cdot\|'$ there is a constant $C > 0$ such that $\|g\|' \leq C\|g\| + C$ for all $g \in G$.

Definition A.6 ([52, Prop. 2.15]). Let G be a Polish group. A subset $A \subseteq G$ is **coarsely bounded** if for every continuous isometric action of G on a metric space (M, d_M), the set $A \cdot m$ is bounded for each $m \in M$, i.e., there is $K > 0$ such that

$$d_M(a_1 \cdot m, a_2 \cdot m) \leq K \quad \text{for all } a_1, a_2 \in A.$$

A Polish group G is **boundedly generated** if it is generated by a coarsely bounded set.

Theorem A.7 ([52, Thm. 2.73]). *A Polish group admits a maximal compatible norm if and only if it is boundedly generated.*

The following characterization is available to establish maximality of a given norm.

Definition A.8 ([52, Def. 2.62]). A norm $\|\cdot\|$ on a group G is called **large-scale geodesic** if there is $K > 0$ such that for any $g \in G$, there are $g_1, \ldots, g_n \in G$ of norm

$\|g_i\| \leq K$, $1 \leq i \leq n$, such that $g = g_1 \cdots g_n$ and

$$\sum_{i=1}^{n} \|g_i\| \leq K \|g\| .$$

Definition A.9 ([52, Lem. 2.39(2) and Prop. 2.7(5)]). A norm $\|\cdot\|$ on a group G is called **coarsely proper** if for every $\epsilon > 0$ and every $R > 0$, there are a finite subset $F \subseteq G$ and $n \in \mathbb{N}$ such that every element $g \in G$ of norm at most R can be written as a product

$$g = f_1 g_1 \cdots f_n g_n,$$

where $f_1, \ldots, f_n \in F$ and each g_i has norm at most ϵ.

Proposition A.10 ([52, Prop. 2.72]). *A compatible norm $\|\cdot\|$ on a Polish group G is maximal if and only if it is both large-scale geodesic and coarsely proper.*

Appendix B

Spaces and groups of measurable maps

B.1 L^0 spaces and convergence in measure

In this section, we introduce the topology of convergence in measure for spaces of measurable functions and explore its connections to an L^1-type metric as well as its relationship with the group $\mathrm{Aut}(X, \mu)$.

Definition B.1. Let (X, μ) be a standard probability space, and let Y be a Polish space. The space $\mathrm{L}^0(X, \mu, Y)$, often denoted by $\mathrm{L}^0(X, Y)$ for brevity, consists of equivalence classes of measurable maps $f : X \to Y$, where functions are identified up to null sets. This space is equipped with the **topology of convergence in measure**, which is generated by the sets $\tilde{U}_{\epsilon,A}$ defined as follows: for every measurable subset $A \subseteq X$, every open subset $U \subseteq Y$, and every $\epsilon > 0$,

$$\tilde{U}_{\epsilon,A} = \{f \in \mathrm{L}^0(X, Y) : \mu(f^{-1}(U) \cap A) > \epsilon\}.$$

The topology of convergence in measure is Polish. The justification of this fact is postponed to Proposition B.8. First, we take a brief detour to justify why this topology is appropriately named the topology of convergence in measure.

Lemma B.2. *Let d_Y be a compatible metric on Y. For $f_0 \in \mathrm{L}^0(X, Y)$, a neighborhood basis of f_0 is given by the sets*

$$\big\{f \in \mathrm{L}^0(X, Y) : \mu\big(\{x \in X : d_Y(f(x), f_0(x)) \geq \epsilon\}\big) < \epsilon\big\}, \quad \epsilon > 0.$$

Proof. Fix $\epsilon > 0$ and a dense sequence $(y_n)_n$ in Y. For each n, define

$$A_n = \big\{x \in X : d_Y(f_0(x), y_n) < \tfrac{\epsilon}{2}\big\}.$$

By the density of $(y_n)_n$, we have $X = \bigcup_n A_n$. Consequently, there exists $N \geq 1$ such that $\mu\big(\bigcup_{n<N} A_n\big) > 1 - \frac{\epsilon}{2}$. Without loss of generality, by re-enumerating the sequence $(y_n)_n$ if necessary, we may assume that $\mu(A_n) > 0$ for all $n < N$. Let U^n denote the open ball of radius $\frac{\epsilon}{2}$ centered at y_n, and set

$$\epsilon_n = \min\big\{\tfrac{\mu(A_n)}{2}, \tfrac{\epsilon}{2N}\big\}.$$

For any $f \in \mathrm{L}^0(X, Y)$, the set $B = \{x \in X : d_Y(f_0(x), f(x)) \geq \epsilon\}$ is contained in

$$\Big(X \setminus \bigcup_{n<N} A_n\Big) \cup \bigcup_{n<N} \{x \in A_n : d_Y(f(x), y_n) \geq \tfrac{\epsilon}{2}\},$$

and therefore

$$\begin{aligned}
\mu(B) &\leq \mu\Big(X \setminus \bigcup_{n<N} A_n\Big) + \sum_{n<N} \mu\big(\{x \in A_n : d_Y(f(x), y_n) \geq \tfrac{\epsilon}{2}\}\big) \\
\because \text{ choice of } N &< \frac{\epsilon}{2} + \sum_{n<N} \mu\big(\{x \in A_n : d_Y(f(x), y_n) \geq \tfrac{\epsilon}{2}\}\big) \\
&= \frac{\epsilon}{2} + \sum_{n<N} \mu\big(A_n \setminus f^{-1}(U^n)\big) \\
&= \frac{\epsilon}{2} + \sum_{n<N} \big(\mu(A_n) - \mu(f^{-1}(U^n) \cap A_n)\big).
\end{aligned}$$

If $f \in \tilde{U}^n_{\mu(A_n)-\epsilon_n, A_n}$, then

$$\mu(A_n) - \mu(f^{-1}(U^n) \cap A_n) < \epsilon_n \leq \tfrac{\epsilon}{2N},$$

and therefore the open set $\bigcap_{n<N} \tilde{U}^n_{\mu(A_n)-\epsilon_n, A_n}$ satisfies the desired inclusion

$$f_0 \in \bigcap_{n<N} \tilde{U}^n_{\mu(A_n)-\epsilon_n, A_n} \subseteq \big\{f \in \mathrm{L}^0(X, Y) : \mu\big(\{x \in X : d_Y(f(x), f_0(x)) \geq \epsilon\}\big) < \epsilon\big\}.$$

Conversely, given any $f_0 \in \tilde{U}_{\epsilon, A}$, we need to show the existence of $\delta > 0$ such that

$$\big\{f \in \mathrm{L}^0(X, Y) : \mu\left(\{x \in X : d_Y(f(x), f_0(x)) \geq \delta\}\right) < \delta\big\} \subseteq U_{\epsilon, A}.$$

Since U is open, we have

$$f_0^{-1}(U) \cap A = \bigcup_n \big\{x \in A : d_Y(f_0(x), Y \setminus U) > \tfrac{1}{n}\big\}.$$

Thus, there exists $n \in \mathbb{N}$ sufficiently large such that

$$\mu\big(\{x \in A : d_Y(f_0(x), Y \setminus U) > \tfrac{1}{n}\}\big) > \epsilon.$$

Let

$$\delta = \min\big\{\tfrac{1}{n}, \mu\big(\{x \in A : d_Y(f_0(x), Y \setminus U) > \tfrac{1}{n}\}\big) - \epsilon\big\}. \tag{B.1}$$

Now, consider $f \in \mathrm{L}^0(X, Y)$ satisfying

$$\mu\left(\{x \in X : d_Y(f_0(x), f(x)) \geq \delta\}\right) < \delta. \tag{B.2}$$

Then,

$$f^{-1}(U) \cap A \supseteq \big\{x \in A : d_Y(f_0(x), f(x)) < \delta \text{ and } d_Y(f_0(x), Y \setminus U) > \tfrac{1}{n}\big\},$$

and the latter set can be rewritten as

$$A \setminus \big(\{x \in A : d_Y(f_0(x), f(x)) \geq \delta\} \cup \{x \in A : d_Y(f_0(x), Y \setminus U) \leq \tfrac{1}{n}\}\big).$$

Consequently,

$$\mu(f^{-1}(U) \cap A) \geq \mu(A) - \mu\big(\{x \in A : d_Y(f_0(x), f(x)) \geq \delta\}\big) \\ - \mu\big(\{x \in A : d_Y(f_0(x), Y \setminus U) \leq \tfrac{1}{n}\}\big).$$

Since f is assumed to satisfy Eq. (B.2) and δ is chosen according to Eq. (B.1), we get

$$\mu(f^{-1}(U) \cap A) > \mu(A) - \mu\big(\{x \in A : d_Y(f_0(x), Y \setminus U) > \tfrac{1}{n}\}\big) + \epsilon \\ - \mu\big(\{x \in A : d_Y(f_0(x), Y \setminus U) \leq \tfrac{1}{n}\}\big).$$

The two negative terms sum to $-\mu(A)$, and thus we have $\mu(f^{-1}(U) \cap A) > \epsilon$. We conclude that

$$\big\{f \in L^0(X, Y) : \mu(\{x \in X : d_Y(f_0(x), f(x)) \geq \delta\}) < \delta\big\} \subseteq \tilde{U}_{\epsilon,A}. \qquad \blacksquare$$

Now suppose that d_Y is a compatible *bounded* metric on Y; for instance,

$$d_Y(y_1, y_2) = \min\{1, d'_Y(y_1, y_2)\}, \quad y_1, y_2 \in Y,$$

for an arbitrary compatible metric d'_Y. We can then equip $L^0(X, Y)$ with the metric $\tilde{d}_Y$, defined by

$$\tilde{d}_Y(f, g) = \int_X d_Y(f(x), g(x))\, d\mu(x). \tag{B.3}$$

The following properties of convergence in measure and $\tilde{d}_Y$ are well-known.

Lemma B.3. *Let d_Y be a compatible bounded metric on Y. The following properties hold:*

(1) *The metric $\tilde{d}_Y$ is compatible with the topology of convergence in measure.*

(2) *A sequence of functions $(f_n)_n$ converges in measure to f if and only if every subsequence of $(f_n)_n$ has a further subsequence that converges pointwise to f.*

Proof. (1) We employ the neighborhood basis established in Lemma B.2. For any $\epsilon > 0$, Markov's inequality yields

$$\epsilon \cdot \mu(\{x \in X : d_Y(f(x), f_0(x)) \geq \epsilon\}) \leq \int_X d_Y(f(x), f_0(x))\, d\mu(x) = \tilde{d}_Y(f, f_0),$$

demonstrating that the topology induced by $\tilde{d}_Y$ refines the topology of convergence in measure.

Conversely, let $K > 0$ be a bound on d_Y. If

$$\mu(\{x \in X : d_Y(f(x), f_0(x)) \geq \frac{\epsilon}{K}\}) < \frac{\epsilon}{K},$$

then $\tilde{d}_Y(f, f_0) < \frac{\epsilon}{K} + \epsilon$. This shows that the topology of convergence in measure refines the topology induced by $\tilde{d}_Y$. Consequently, the two topologies coincide.

(2) We begin with the direct implication. Assume that $f_n \to f$ in measure and consider a subsequence $(f_{n_k})_k$. By passing to a further subsequence (still denoted $(f_{n_k})_k$ for simplicity), Lemma B.2 lets us assume that for all $k \in \mathbb{N}$, the set

$$A_k = \{x \in X : d_Y(f_{n_k}(x), f(x)) \geq 2^{-k}\}$$

has measure less than 2^{-k}. By the Borel–Cantelli lemma, almost every $x \in X$ belongs to only finitely many A_k. This implies that f_{n_k} converges pointwise to f almost surely.

For the converse, assume that every subsequence of $(f_n)_n$ admits a further subsequence that converges pointwise to f. Note that $\tilde{d}_Y(f_n, f) \to 0$ holds if and only if every subsequence $(f_{n_k})_k$ of $(f_n)_n$ has a further subsequence $(f_{n_{k_i}})_i$ such that $\tilde{d}_Y(f_{n_{k_i}}, f) \to 0$. Thus, it suffices to show that if $(f_n)_n$ converges to f pointwise, then also $\tilde{d}_Y(f_n, f) \to 0$. This follows directly from the Lebesgue's dominated convergence theorem, in view of the boundedness of d_Y. ■

We finish this section with the following lemma.

Lemma B.4. *Let G be a Polish group acting continuously on a Polish space X, and let μ be a Borel probability measure on X. Then the map*

$$\begin{aligned} \Phi : \mathrm{L}^0(X, G) &\to \mathrm{L}^0(X, X) \\ \mathrm{L}^0(X, G) \ni f &\mapsto (x \mapsto f(x) \cdot x) \end{aligned}$$

is continuous.

Proof. We use the pointwise characterization of convergence in measure, as stated in item (2) of Lemma B.3. Suppose $f_n \to f$ in measure, where $f_n, f \in \mathrm{L}^0(X, G)$. To show that $\Phi(f_n) \to \Phi(f)$ in measure, consider an arbitrary subsequence $(\Phi(f_{n_k}))_k$.

Since $f_n \to f$ in measure, there exists a further subsequence $(f_{n_{k_i}})_i$ that converges pointwise to f. By the continuity of the action, we have $f_{n_{k_i}}(x) \cdot x \to f(x) \cdot x$ for all $x \in X$. This means $(\Phi(f_{n_{k_i}}))_i$ converges pointwise to $\Phi(f)$, as required. ■

B.2 L^1 spaces of pointed metric spaces

We now restrict our attention to integrable maps. To this end, we require the target space to be a **Polish pointed metric space**, which we define as a separable complete metric space (Y, d_Y) equipped with a distinguished basepoint $e \in Y$.

Definition B.5. Let (X, μ) be a standard probability space, and let (Y, d_Y, e) be a Polish pointed metric space. We define the ***e*-pointed L^1 space** $\mathrm{L}^1_e(X, \mu, Y)$, often denoted by

$\mathrm{L}^1_e(X,Y)$ for brevity, as the pointed metric space consisting of all measurable functions $f : X \to Y$ satisfying

$$\int_X d_Y(e, f(x))\, d\mu(x) < +\infty,$$

equipped with the metric

$$\tilde{d}_Y(f_1, f_2) = \int_X d_Y\big(f_1(x), f_2(x)\big)\, d\mu(x),$$

and with the constant function $\hat{e} : x \mapsto e$ as its basepoint. The finiteness of the integral in the definition of $\tilde{d}_Y$ follows from the triangle inequality, using $\hat{e}$ as an intermediate point.

Remark B.6. To simplify notation, we omit explicit reference to the metric d_Y in the notation for the L^1 space $\mathrm{L}^1_e(X,Y)$. However, it is important to note that the definition of this space fundamentally depends on the choice of d_Y.

Proposition B.7. *Let (X,μ) be a standard probability space and (Y, d_Y, e) be a Polish pointed metric space. Then $(\mathrm{L}^1_e(X,Y), \tilde{d}_Y)$ is a Polish metric space.*

Proof. The proof follows the classical argument establishing that $(\mathrm{L}^1(X,\mathbb{R}), \tilde{d}_{\mathbb{R}})$ is a Polish metric space. To prove completeness, consider a Cauchy sequence $(f_n)_n$ in $\mathrm{L}^1_e(X,Y)$. Without loss of generality, we may assume that $\tilde{d}_Y(f_n, f_{n+1}) < 2^{-n}$, $n \in \mathbb{N}$. Define the sets

$$A_n = \{x \in X : d_Y(f_n(x), f_{n+1}(x)) \geq 1/n^2\}, \quad n \geq 1.$$

By Markov's inequality, $\mu(A_n) \leq n^2 2^{-n}$, and thus $\sum_n \mu(A_n) < \infty$. The Borel–Cantelli lemma implies that $(f_n(x))_n$ is pointwise Cauchy for almost every $x \in X$. Since (Y, d_Y) is complete, the pointwise limit $f(x) = \lim_n f_n(x)$ exists almost surely.

Define functions $h_n, h : X \to \mathbb{R}^{\geq 0}$ by

$$h_n(x) = \sum_{i<n} d_Y(f_i(x), f_{i+1}(x)), \quad h(x) = \sum_{i \in \mathbb{N}} d_Y(f_i(x), f_{i+1}(x)) = \lim_{n\to\infty} h_n(x),$$

and note that $h \in \mathrm{L}^1(X,\mathbb{R})$ by Fatou's lemma. Finally, we conclude that

$$\begin{aligned}\tilde{d}_Y(f_n, f) = \int_X d_Y(f_n(x), f(x))\, d\mu(x) &\leq \int_X \sum_{k=n}^{\infty} d_Y(f_k(x), f_{k+1}(x))\, d\mu(x) \\ &= \int_X (h(x) - h_n(x))\, d\mu(x) \to 0,\end{aligned}$$

where the last convergence follows from Lebesgue's dominated convergence theorem.

To establish separability, let $D \subseteq Y$ be a countable dense set. The subspace of maps taking values in D is $\tilde{d}_Y$-dense (in fact, dense even in the much stronger *sup* metric).

The set of functions taking only finitely many values in D remains dense. By further restricting to functions measurable with respect to a dense countable subalgebra of the measure algebra on X, we obtain a countable dense subset of $\mathrm{L}^1_e(X,Y)$. ∎

Note that $\mathrm{L}^1_e(X,Y)$ is a subset of $\mathrm{L}^0(X,Y)$. If d_Y is bounded, the integrability condition becomes trivial, and we have $\mathrm{L}^1_e(X,Y) = \mathrm{L}^0(X,Y)$. Combining item (1) from Lemma B.3 with the preceding proposition, we obtain the following well-known result.

Proposition B.8. *If Y is a Polish space, then the topology of convergence in measure on $\mathrm{L}^0(X,Y)$ is Polish, and it is induced by the L^1 metric $\tilde{d}_Y$ for any bounded compatible metric d_Y on Y.* ∎

More generally, we can establish a relationship between the L^1 topology and the topology of convergence in measure as follows.

Proposition B.9. *Let (Y, d_Y, e) be a possibly unbounded Polish pointed metric space. The inclusion map $\mathrm{L}^1_e(X,Y) \hookrightarrow \mathrm{L}^0(X,Y)$ is continuous.*

Proof. Define $d^b_Y(y_1,y_2) = \min\{d_Y(y_1,y_2),1\}$, $y_1,y_2 \in Y$, to be the bounded complete metric on Y obtained by capping d_Y. By Lemma B.3, $\tilde{d}^b_Y$ induces the topology of convergence in measure on $\mathrm{L}^0(X,Y)$. Since $d^b_Y \leq d_Y$, one also has $\tilde{d}^b_Y \leq \tilde{d}_Y$. Thus, the inclusion map

$$(\mathrm{L}^1_e(X,Y), \tilde{d}_Y) \hookrightarrow (\mathrm{L}^0(X,Y), \tilde{d}^b_Y)$$

is 1-Lipschitz and, in particular, continuous. ∎

Remark B.10. The inclusion $\mathrm{L}^1_e(X,Y) \hookrightarrow \mathrm{L}^0(X,Y)$ is not, in general, an embedding. To see this, suppose d_Y is unbounded. Then there exist elements $y_n \in Y$ such that $d_Y(y_n, e) \geq 2^n$ for all n. Partition the space X into disjoint sets $\bigsqcup_{n=1}^{\infty} A_n$, where $\mu(A_n) = 2^{-n}$. Define functions f_n for $n \geq 1$ as follows:

$$f_n(x) = \begin{cases} y_n & \text{if } x \in A_n, \\ e & \text{otherwise.} \end{cases}$$

In the notation of Proposition B.9, we have $\tilde{d}^b_Y(f_n, \hat{e}) = \mu(A_n) \to 0$ as $n \to \infty$, which implies $f_n \to \hat{e}$ in L^0. However, since $\tilde{d}_Y(f_n, \hat{e}) \geq 1$ for all $n \geq 1$, this convergence does not hold in L^1.

The group of measure-preserving automorphisms $\mathrm{Aut}(X,\mu)$ acts naturally on $\mathrm{L}^1_e(X,Y)$ by composition, i.e., $(T \cdot f)(x) = f(T^{-1}x)$. This action fixes the basepoint $\hat{e}$. Moreover, each automorphism acts by an isometry, as it preserves the measure μ.

Proposition B.11. *Let (X,μ) be a standard probability space, and let (Y, d_Y, e) be a Polish pointed metric space. The action of $\mathrm{Aut}(X,\mu)$ on $\mathrm{L}^1_e(X,Y)$ is continuous.*

Proof. The argument follows a similar approach to that in [11, Prop. 2.9(1)]. Consider sequences $T_n \to T$ and $f_n \to f$. We need to show that $T_n \cdot f_n \to T \cdot f$. Since the action is by isometries, we have

$$\tilde{d}_Y(T_n \cdot f_n, T \cdot f) = \tilde{d}_Y(f_n, T_n^{-1}T \cdot f) \leq \tilde{d}_Y(f_n, f) + \tilde{d}_Y(f, T_n^{-1}T \cdot f).$$

Thus, it suffices to prove that for any $f \in \mathrm{L}^1_e(X, Y)$ and any convergent sequence of automorphisms $T_n \to T$, the term $\tilde{d}_Y(f, T_n^{-1}T \cdot f)$ tends to 0 as $n \to \infty$.

To establish this, it is enough to verify the claim for functions that take only finitely many values, as such functions are dense in $\mathrm{L}^1_e(X, Y)$. Suppose f is a step function defined over a partition $X = \bigsqcup_{i=1}^m A_i$. The convergence $T_n \to T$ implies $\mu(T_n^{-1}T(A_i) \triangle A_i) \to 0$ for all $1 \leq i \leq m$, which readily gives $\tilde{d}_Y(f, T_n^{-1}T \cdot f) \to 0$. ■

In what follows, we identify $\mathrm{Aut}(X, \mu)$ with a subset of $\mathrm{L}^0(X, X)$.

Proposition B.12. *Let τ be a Polish topology on a standard probability space (X, μ) compatible with its Borel structure. The inclusion map*

$$\mathrm{Aut}(X, \mu) \hookrightarrow \mathrm{L}^0(X, X)$$

is a topological embedding when $\mathrm{Aut}(X, \mu)$ *is equipped with the weak topology and* $\mathrm{L}^0(X, X)$ *is equipped with the topology of convergence in measure associated with τ.*

Proof. Fix a bounded complete metric d compatible with τ. By Lemma B.3, $\tilde{d}$ induces the topology of convergence in measure on $\mathrm{L}^0(X, X)$. The topological group $\mathrm{Aut}(X, \mu)$ acts on $\mathrm{L}^0(X, X)$ by $T \cdot f = f \circ T^{-1}$ and this action is continuous by Proposition B.11. Furthermore, for every $T \in \mathrm{Aut}(X, \mu)$, we have $T^{-1} \cdot \mathrm{id}_X = T \in \mathrm{L}^0(X, X)$, which shows that the inclusion map is continuous.

The metric $\tilde{d}$ induces a right-invariant metric on $\mathrm{Aut}(X, \mu)$. To prove that the inclusion map is a topological embedding, it suffices to show that if $\tilde{d}(T_n, \mathrm{id}_X) \to 0$, then $T_n \to \mathrm{id}_X$ weakly in $\mathrm{Aut}(X, \mu)$. By Lemma B.3, $\tilde{d}(T_n, \mathrm{id}_X) \to 0$ implies $T_n \to \mathrm{id}_X$ in measure. In particular, for every τ-open set $U \subseteq X$, the definition of convergence in measure yields

$$\liminf_{n \to \infty} \mu(T_n^{-1}(U) \cap U) \geq \mu(\mathrm{id}_X^{-1}(U) \cap U) = \mu(U).$$

We conclude that

$$\mu(U) \geq \limsup_{n \to \infty} \mu(T_n^{-1}(U) \cap U) \geq \liminf_{n \to \infty} \mu(T_n^{-1}(U) \cap U) \geq \mu(U),$$

and therefore $\lim_n \mu(T_n^{-1}(U) \cap U) = \mu(U)$. Since each T_n is measure-preserving, it follows that $\lim_n \mu(T_n^{-1}(U) \triangle U) = 0$ for every $U \in \tau$. By the regularity of μ, this implies $T_n \to \mathrm{id}_X$ weakly, as required. ■

Let us now return to L^1 spaces associated with possibly unbounded pointed metric spaces (Y, d_Y, e). When Y is a Polish group, there is a natural choice for e, namely the identity element of the group, which we also denote by e. In this case, we simplify the notation further and write $\mathrm{L}^1(X, Y)$.

Recall that a **Polish normed group** is a Polish group equipped with a compatible norm (see Appendix A.1). In particular, if $(G, \|\cdot\|)$ is a Polish normed group, there is a canonical choice of a compatible complete metric on G, namely

$$d_G(u, v) = \frac{\|u^{-1}v\| + \|vu^{-1}\|}{2}.$$

The corresponding space $\mathrm{L}^1(X, G)$ is Polish by Proposition B.7. Furthermore, it forms a Polish group under pointwise operations, as we now demonstrate.

Proposition B.13. *Let $(G, \|\cdot\|)$ be a Polish normed group. The space $\mathrm{L}^1(X, G)$ is a Polish normed group under the pointwise operations,*

$$(f \cdot g)(x) = f(x)g(x), \quad f^{-1}(x) = f(x)^{-1},$$

and the norm $\|f\|_1^{\mathrm{L}^1(X,G)} = \int_X \|f(x)\| \, d\mu(x)$.

Proof. The space $\mathrm{L}^1(X, G)$ consists of all measurable functions $f : X \to G$ with finite norm, $\|f\|_1^{\mathrm{L}^1(X,G)} < \infty$. Using the properties of the norm $\|\cdot\|$ on G, we have

$$\begin{aligned} \|f \cdot g\|_1^{\mathrm{L}^1(X,G)} &= \int_X \|f(x)g(x)\| \, d\mu(x) \\ &\le \int_X \big(\|f(x)\| + \|g(x)\|\big) \, d\mu(x) \\ &= \|f\|_1^{\mathrm{L}^1(X,G)} + \|g\|_1^{\mathrm{L}^1(X,G)}, \\ \|f^{-1}\|_1^{\mathrm{L}^1(X,G)} &= \int_X \|f(x)^{-1}\| \, d\mu(x) \\ &= \int_X \|f(x)\| \, d\mu(x) = \|f\|_1^{\mathrm{L}^1(X,G)}. \end{aligned}$$

Thus, $\mathrm{L}^1(X, G)$ is closed under the group operations, and $\|\cdot\|_1^{\mathrm{L}^1(X,G)}$ defines a group norm on it.

To verify the continuity of the group operations, it suffices to show that for any $g \in \mathrm{L}^1(X, G)$ and any sequence $f_n \in \mathrm{L}^1(X, G)$, $n \in \mathbb{N}$, converging to the identity function $\hat{e}$ (i.e., $\|f_n\|_1^{\mathrm{L}^1(X,G)} \to 0$), there exists a subsequence $(f_{n_k})_k$ such that

$$\|g \cdot f_{n_k} \cdot g^{-1}\|_1^{\mathrm{L}^1(X,G)} \to 0 \text{ as } k \to \infty,$$

(see, for example, [9, Thm 3.4 and Lem. 3.5]).

By Proposition B.9 and the fact that convergence in measure implies pointwise convergence of a subsequence (see item (2) of Lemma B.3), there exists a subsequence $(f_{n_k})_k$ such that $f_{n_k}(x) \to e$ for almost all $x \in X$. By passing to a further subsequence, we may assume without loss of generality that $\sum_k \|f_{n_k}\|_1^{\mathrm{L}^1(X,G)} < +\infty$. The function $M(x) = \sum_k \|f_{n_k}(x)\|$ belongs to $\mathrm{L}^1(X,G)$, and for all $k \in \mathbb{N}$ and $x \in X$,

$$\|g(x) f_{n_k}(x) g(x)^{-1}\| \leq 2\|g(x)\| + \|f_{n_k}(x)\| \leq 2\|g(x)\| + M(x).$$

The continuity of the group operations on G ensures that $g \cdot f_{n_k} \cdot g^{-1} \to \hat{e}$ pointwise. It remains to apply Lebesgue's dominated convergence theorem and conclude that $\|g \cdot f_{n_k} \cdot g^{-1}\|_1^{\mathrm{L}^1(X,G)} \to 0$, as required. ∎

Remark B.14. If the chosen compatible norm on G is bounded, then $\mathrm{L}^1(X,G) = \mathrm{L}^0(X,G)$, and the topology under consideration coincides with the topology of convergence in measure by Lemma B.3. Consequently, we recover the well-known fact that $\mathrm{L}^0(X,G)$ is a Polish group when equipped with the topology of convergence in measure.

Appendix C

Hopf decomposition

An important tool in the theory of invertible non-singular transformations on σ-finite measure spaces is the Hopf decomposition, which partitions the phase space into the so-called *dissipative* and *recurrent* parts, reflecting different dynamics of the transformation. In this appendix, we recall the relevant definitions and indicate what happens for measure-preserving transformations of a σ-finite space. The reader may consult [35, Sec. 1.3] for further details on the following definitions.

Definition C.1. Let S be an invertible non-singular transformation of a σ-finite measure space (Ω, λ). A measurable set $A \subseteq \Omega$ is said to be:

- **wandering** if $A \cap S^k(A) = \varnothing$ for all $k \geq 1$;
- **recurrent** if $A \subseteq \bigcup_{k\geq 1} S^k(A)$;
- **infinitely recurrent** if $A \subseteq \bigcap_{n\geq 1} \bigcup_{k\geq n} S^k(A)$.

The inclusions above are understood to hold up to a null set. The transformation S is:

- **dissipative** if the phase space Ω is a countable union of wandering sets;
- **conservative** if there are no wandering sets of positive measure;
- **recurrent** if every set of positive measure is recurrent;
- **infinitely recurrent** if every set of positive measure is infinitely recurrent.

It turns out that the properties of being conservative, recurrent, and infinitely recurrent are all mutually equivalent.

Proposition C.2. *Let S be an invertible non-singular transformation of a σ-finite measure space (Ω, λ). The following are equivalent:*

(1) *S is conservative;*

(2) *S is recurrent;*

(3) *S is infinitely recurrent.*

Among the properties introduced in Definition C.1, only recurrence and dissipativity are therefore different. In fact, any non-singular transformation admits a canonical decomposition, known as the Hopf decomposition, into these two types of action.

Proposition C.3 (Hopf decomposition). *Let S be an invertible non-singular transformation of a σ-finite measure space (Ω, λ). There exists an S-invariant partition $\Omega = D \sqcup C$ such that $S\restriction_D$ is dissipative and $S\restriction_C$ is recurrent (equivalently, conserva-*

tive). Moreover, if $\Omega = D' \sqcup C'$ *is another partition with this property then* $\lambda(D \triangle D') = 0$ *and* $\lambda(C \triangle C') = 0$.

We also note the following consequence of dissipativity in case the measure is preserved.

Proposition C.4. *Let* S *be an invertible measure-preserving transformation of a* σ*-finite measure space* (Ω, λ) *and let* $\Omega = D \sqcup C$ *be its Hopf decomposition. For every set* $A \subseteq \Omega$ *of finite measure, almost every point in* D *eventually escapes* A*:*

$$\forall^\lambda x \in D\ \exists N\ \forall n \geq N\ T^n x \notin A.$$

Proof. We may as well assume $D = \Omega$. Let $A \subseteq \Omega$ have finite measure. Let Q be a wandering set whose translates cover Ω as provided by [1, Prop. 1.1.2]. Consider the map $\Phi : Q \times \mathbb{Z} \to \Omega$ which maps (x, n) to $T^n(x)$, and observe that Φ is measure-preserving if we endow $Q \times \mathbb{Z}$ with the product of the measure induced by λ on Q and the counting measure on $\mathbb{Z}$.

If the set of $x \in Q$ for which $S^n(x) \in A$ for infinitely many $n \in \mathbb{N}$ has positive measure, then, by Fubini's theorem, the set A must have infinite measure, which contradicts the assumption. The same reasoning applies to any S-translate of Q. Since these translates cover Ω, the proof is complete. ■

Appendix D

Disintegration of measure

Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space (X, μ), and let $\pi : X \to Y$ be a measurable reduction to the identity relation on some standard Lebesgue space (Y, ν), $\pi(x) = \pi(y)$ if and only if $x\mathcal{R}y$. Suppose that ν is a σ-finite measure on Y that is equivalent to the push-forward $\pi_*\mu$. A **disintegration** of μ relative to (π, ν) is a collection of measures $(\mu_y)_{y\in Y}$ on X such that for all Borel sets $A \subseteq X$:

(1) $\mu_y(X \setminus \pi^{-1}(y)) = 0$ for ν-almost all $y \in Y$;

(2) the map $Y \ni y \mapsto \mu_y(A) \in \mathbb{R}$ is measurable;

(3) $\mu(A) = \int_Y \mu_y(A)\, d\nu(y)$.

A theorem of D. Maharam [43] asserts that μ can be disintegrated over any (π, ν) as above. In fact, the existence of a disintegration can be proved in a considerably more general setup (see, for example, D. H. Fremlin [19, Thm. 452I]), but in the framework of standard Lebesgue spaces, disintegration is essentially unique. While the context of our work is purely ergodic-theoretical, we note that the disintegration result holds in the descriptive set-theoretical setting as well, as discussed in [44] and [26]. Without striving for generality, we formulate here a specific version that suits our needs.

Theorem D.1 (Disintegration of Measure). *Let (X, μ) be a standard Lebesgue space, (Y, ν) be a σ-finite standard Lebesgue space, and let $\pi : X \to Y$ be a measurable function. If $\pi_*\mu$ is equivalent to ν, then there exists a disintegration $(\mu_y)_{y\in Y}$ of μ over (π, ν). Moreover, such a disintegration is essentially unique in the sense that if $(\mu'_y)_{y\in Y}$ is another disintegration, then $\mu_y = \mu'_y$ for ν-almost all $y \in Y$.*

Remark D.2. It is more common to formulate the disintegration theorem with the assumption that $\pi_*\mu = \nu$, when one can additionally ensure that $\mu_y(X) = \mu(X)$ for ν-almost all y. Weakening the equality $\pi_*\mu = \nu$ to mere equivalence is a simple consequence, for if $(\mu_y)_{y\in Y}$ is a disintegration of μ over $(\pi, \pi_*\mu)$, then $\big(\frac{d\pi_*\mu}{d\nu}(y) \cdot \mu_y\big)_{y\in Y}$ is a disintegration of μ over (π, ν).

Let $X_a \subseteq X$ be the set of atoms of the disintegration, i.e.,

$$X_a = \{x \in X : \mu_y(x) > 0 \text{ for some } y \in Y\},$$

and let F be the equivalence relation on X_a, where two atoms within the same fiber are equivalent whenever they have the same measure: $x_1 F x_2$ if and only if $\mu_{\pi(x_1)}(x_1) = \mu_{\pi(x_2)}(x_2)$ and $\pi(x_1) = \pi(x_2)$. The equivalence relation F is measurable and has finite classes μ-almost surely. Let X_n, $n \geq 1$, be the union of F-equivalence classes of size exactly n, thus $X_a = \bigsqcup_{n\geq 1} X_n$. Set also $X_0 = X \setminus X_a$ to be the atomless part of the disintegration and let $\mathcal{R}_n$ denote the restriction of $\mathcal{R}$ onto X_n.

Consider the group $[\mathcal{R}] \leq \mathrm{Aut}(X, \mu)$ of measure-preserving bijections T such that $x\mathcal{R}Tx$ holds μ-almost surely. Every $T \in [\mathcal{R}]$ preserves ν-almost all measures μ_y, since $(T_*\mu_y)_{y\in Y}$ is a disintegration of $T_*\mu = \mu$, which has to coincide with $(\mu_y)_{y\in Y}$ by the uniqueness of the disintegration. In particular, the partition $X = \bigsqcup_{n\in\mathbb{N}} X_n$ is invariant under the full group $[\mathcal{R}]$, and for any $T \in [\mathcal{R}]$, the restriction $T \restriction_{X_n} \in [\mathcal{R}_n]$ for every $n \in \mathbb{N}$. Conversely, for a sequence $T_n \in [\mathcal{R}_n]$, $n \in \mathbb{N}$, one has $T = \bigsqcup_n T_n \in [\mathcal{R}]$. We therefore have an isomorphism of (abstract) groups $[\mathcal{R}] \cong \prod_{n\in\mathbb{N}}[\mathcal{R}_n]$.

The groups $[\mathcal{R}_n]$ can be described quite explicitly. First, consider the case $n \geq 1$; thus $X_n \subseteq X_a$. All equivalence classes of the restriction of F onto X_n have size n. Let $Y_n \subseteq X_n$ be a measurable transversal, i.e., a measurable set intersecting every F-class in a single point, and let $\nu_n = \mu \restriction_{Y_n}$. Every $T \in [\mathcal{R}_n]$ produces a permutation of μ-almost every F-class, so we can view it as an element of $\mathrm{L}^0(Y_n, \nu_n, \mathfrak{S}_n)$, where $\mathfrak{S}_n$ is the group of permutations of an n-element set. This identification works in both directions and produces an isomorphism $[\mathcal{R}_n] \cong \mathrm{L}^0(Y_n, \nu_n, \mathfrak{S}_n)$. Note also that all ν_n are atomless if so is μ. We allow for $\mu(X_n) = 0$, in which case $\mathrm{L}^0(Y_n, \nu_n, \mathfrak{S}_n)$ is the trivial group.

Let us now return to the equivalence relation $\mathcal{R}_0 = \mathcal{R} \cap X_0 \times X_0$, and recall that the measures $\mu_y \restriction_{X_0}$ are atomless. Let $Y_0 = \{y : \mu_y(X_0) > 0\}$ be the encoding of fibers with non-trivial atomless components, and put $\nu_0 = \nu \restriction_{Y_0}$. In particular, for every $y \in Y_0$, the space (X_0, μ_y) is isomorphic to the interval $[0, \mu_y(X_0)]$ endowed with the Lebesgue measure. In fact, one can select such isomorphisms in a measurable way across all $y \in Y_0$. More precisely, there is a measurable isomorphism

$$\psi : X_0 \to \{(y, r) \in Y_0 \times \mathbb{R} : 0 \leq r \leq \mu_y(X_0)\}$$

such that for all $y \in Y_0$,

- $\psi(\pi^{-1}(y) \cap X_0) = \{y\} \times [0, \mu_y(X_0)]$;
- $\psi_*(\mu_y \restriction_{X_0})$ coincides with the Lebesgue measure on $\{y\} \times [0, \mu_y(X_0)]$.

The reader may find further details in [26, Thm. 2.3], where the same construction is discussed in a more refined setting of Borel disintegrations.

Using the isomorphism ψ, each $\pi^{-1}(y) \cap X_0$, $y \in Y_0$, is identified with $[0, \mu_y(X_0)]$. Since every $T \in [\mathcal{R}_0]$ preserves ν-almost every μ_y, we may rescale these intervals and view any $T \in [\mathcal{R}_0]$ as an element of $\mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0, 1], \lambda))$. Conversely, every $f \in \mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0, 1], \lambda))$ gives rise to $T_f \in [\mathcal{R}_0]$ via the notationally convoluted but natural

$$T_f(x) = \psi^{-1}\big(\pi(x), \big(f(\pi(x)) \cdot \mathrm{proj}_2(\psi(x))/\mu_{\pi(x)}(X_0)\big)\mu_{\pi(x)}(X_0)\big),$$

which, in plain words, simply applies $f(\pi(x))$ upon the corresponding fiber identified with $[0, 1]$ using ψ. This map is an isomorphism between the groups $[\mathcal{R}_0]$ and $\mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0, 1], \lambda))$.

Let us say that $\mathcal{R}$ has **atomless classes** if μ_y is atomless ν-almost surely or, equivalently, if $\mu(X_a) = 0$ in the notation above. We may summarize the discussion so far into the following proposition.

Proposition D.3. *Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space (X, μ). There are (possibly empty) standard Lebesgue spaces (Y_n, ν_n), $n \in \mathbb{N}$, such that the full group $[\mathcal{R}] \leq \mathrm{Aut}(X, \mu)$ is (abstractly) isomorphic to*

$$\mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0,1], \lambda)) \times \prod_{n \geq 1} \mathrm{L}^0(Y_n, \nu_n, \mathfrak{S}_n),$$

where $\mathfrak{S}_n$ is the group of permutations of an n-element set. If μ is atomless, then so are the spaces (Y_n, ν_n), $n \geq 1$. If $\mathcal{R}$ has atomless classes, then all (Y_n, ν_n), $n \geq 1$, are negligible and $[\mathcal{R}]$ is isomorphic to $\mathrm{L}^0(Y_0, \nu_0, \mathrm{Aut}([0,1], \lambda))$.

We can further refine the product in Proposition D.3 by decomposing the spaces (Y_n, ν_n) into individual atoms and the atomless remainders. This relies on the following general result.

Proposition D.4. *Let (Z, ν_Z) be a standard Lebesgue space, and let G be a Polish group. For any finite or countably infinite measurable partition $Z = \bigsqcup_{n \in I} Z_n$, there exists an isomorphism of topological groups between $\mathrm{L}^0(Z, \nu_Z, G)$ and $\prod_{n \in I} \mathrm{L}^0(Z_n, \nu_{Z,n}, G)$, where $\nu_{Z,n}$ is the restriction of ν_Z onto Z_n.*

Proof. Consider the map Φ that assigns to each $f \in \mathrm{L}^0(Z, \nu_Z, G)$ the sequence of its restrictions $f \restriction_{Z_n} \in \mathrm{L}^0(Z_n, \nu_{Z,n}, G)$, where $n \in I$. It is straightforward to verify that Φ is a group isomorphism, and its continuity follows directly from the definition of convergence in measure. Automatic continuity implies that Φ is a homeomorphism, as it is a Borel group isomorphism between Polish groups (see [6, Sec. 1.6]). Alternatively, the continuity of Φ^{-1} can easily be checked directly. ∎

Applying Proposition D.4 to the partition of (Z, ν_Z) into the atomless part Z_0 and individual atoms $Z_k = \{z_k\}$ (if any), and noting that for a singleton Z_k the group $\mathrm{L}^0(Z_k, \nu_{Z,k}, G)$ is naturally isomorphic to G, we get the following corollary.

Corollary D.5. *Let (Z, ν_Z) be a standard Lebesgue space, and let G be a Polish group. Let $Z_a \subseteq Z$ be the set of atoms of Z, and let $Z_0 = Z \setminus Z_a$ be the atomless part. The group $\mathrm{L}^0(Z, \nu_Z, G)$ is isomorphic to $\mathrm{L}^0(Z_0, \nu_Z \restriction_{Z_0}, G) \times G^{|Z_a|}$.*

Combining the above discussion with Proposition D.3, we obtain a very concrete representation for $[\mathcal{R}]$. In the formulation below, G^0 is understood to be the trivial group.

Proposition D.6. *Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space (X, μ). There exist cardinals $\kappa_n \leq \aleph_0$ and $\epsilon_n \in \{0, 1\}$ such that*

$$[\mathcal{R}] \cong \mathrm{L}^0([0,1],\lambda,\mathrm{Aut}([0,1],\lambda))^{\epsilon_0} \times \mathrm{Aut}([0,1],\lambda)^{\kappa_0} \times \Big(\prod_{n\geq 1} \mathrm{L}^0([0,1],\lambda,\mathfrak{S}_n)^{\epsilon_n} \times \mathfrak{S}_n^{\kappa_n}\Big).$$

If μ is atomless, then $\kappa_n = 0$ for all $n \geq 1$; if $\mathcal{R}$ has atomless classes, then $\epsilon_n = 0$ for all $n \geq 1$.

So far, we have considered $[\mathcal{R}]$ as an abstract group. This is because neither of the two natural topologies on $\mathrm{Aut}(X,\mu)$ interacts well with the full group construction—$[\mathcal{R}]$ is generally not closed in the weak topology, and is not separable in the uniform topology whenever $\mu(X_0) > 0$. Nonetheless, the isomorphism given in Proposition D.3 shows that there is a natural Polish topology on $[\mathcal{R}]$, which arises when we view the groups $\mathrm{L}^0(Y_0,\nu_0,\mathrm{Aut}([0,1],\lambda))$ and $\mathrm{L}^0(Y_n,\nu_n,\mathfrak{S}_n)$ as Polish groups in the topology of convergence in measure. It is with respect to this topology that we formulate Proposition D.7.

Proposition D.7. *Let $\mathcal{R}$ be a smooth measurable equivalence relation on a standard Lebesgue space (X,μ). The set of periodic elements is dense in $[\mathcal{R}]$.*

Proof. Rokhlin's Lemma implies that any $T \in [\mathcal{R}]$ can be approximated in the uniform topology by periodic elements from $[T] \subseteq [\mathcal{R}]$. Since the uniform topology is stronger than the Polish topology on $[\mathcal{R}]$, the proposition follows. ∎

Appendix E

Actions of locally compact Polish groups

In this chapter of the appendix, we collect some well-known facts related to the actions of locally compact Polish groups. This exposition is provided for the reader's convenience and completeness. We recall that, by a result of G. W. Mackey [42], any Boolean measure-preserving action of a locally compact Polish group can be realized as a spatial Borel action. Thus, we may switch to pointwise formulations whenever convenient for the exposition.

E.1 Ergodic decomposition

Let $G \curvearrowright X$ be a measure-preserving action of a locally compact Polish group on a standard probability space (X, μ). The space $\mathcal{E} = \mathrm{EINV}(G \curvearrowright X)$ of invariant ergodic probability measures of this action possesses the structure of a standard Borel space. The Ergodic Decomposition theorem of V. S. Varadarajan [59, Thm. 4.2] asserts that there exist an essentially unique Borel G-invariant surjection $X \ni x \mapsto \nu_x \in \mathcal{E}$ and a probability measure p on $\mathcal{E}$ such that $\mu = \int_{\mathcal{E}} \nu \, dp(\nu)$. This equality holds in the sense that $\mu(A) = \int_{\mathcal{E}} \nu(A) \, dp(\nu)$ for all Borel $A \subseteq X$.

There is a one-to-one correspondence between measurable G-invariant functions $h : X \to \mathbb{R}$ and measurable functions $\tilde{h} : \mathcal{E} \to \mathbb{R}$, given by $\tilde{h}(\nu_x) = h(x)$. For measures μ and p as above, this correspondence gives an isometric isomorphism between $\mathrm{L}^1(\mathcal{E}, \mathbb{R})$ and the subspace of $\mathrm{L}^1(X, \mathbb{R})$ consisting of G-invariant functions.

E.2 Tessellations

An important feature of locally compact group actions is the fact that they all admit Lebesgue measurable cross-sections. This was proved by J. Feldman, P. Hahn, and C. Moore in [16], whereas a Borel version of the result was obtained by A. S. Kechris in [30].

Definition E.1. Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group. A **cross-section** is a Borel set $C \subseteq X$ which is both

- a **complete section** for $\mathcal{R}_G$: it intersects every orbit of the action; and
- **lacunary**: there is a neighborhood of the identity $1_G \in U \subseteq G$ such that C is U-lacunary, namely one has $U \cdot c \cap U \cdot c' = \varnothing$ for all distinct $c, c' \in C$.

A cross-section C is K-**cocompact**, where $K \subseteq G$ is a compact set, if $K \cdot C = X$; a cross-section is **cocompact** if it is K-cocompact for some compact $K \subseteq G$.

Any action $G \curvearrowright X$ admits a K-cocompact cross-section, whenever $K \subseteq G$ is a compact neighborhood of the identity (see [56, Thm. 2.4]). We also recall the following well-known lemma on the possibility of partitioning a cross-section into pieces with a prescribed lacunarity parameter.

Lemma E.2. *Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group, and let C be a cross-section for the action. For any compact neighborhood of the identity $V \subseteq G$, there exists a finite Borel partition $C = \bigsqcup_i C_i$ such that each C_i is V-lacunary.*

Proof. Set $K = (V \cup V^{-1})^2$, and let $U \subseteq G$ be a compact neighborhood of the identity small enough for C to be U-lacunary. Define a binary relation $\mathcal{G}$ on C by declaring $(c, c') \in \mathcal{G}$ whenever $c \in K \cdot c'$ and $c \neq c'$. Note that $\mathcal{G}$ is symmetric since K is. We view $\mathcal{G}$ as a Borel graph on C and claim that it is locally finite. More specifically, if λ is a right Haar measure, then the degree of each $c \in C$ is at most $\lfloor \frac{\lambda(U \cdot K)}{\lambda(U)} \rfloor - 1$.

Indeed, let $c_0, \ldots, c_N \in C$ be distinct elements such that $c_i \in K \cdot c_0$ for all $i \leq N$; in particular $(c_i, c_0) \in \mathcal{G}$ for $i \geq 1$. Let $k_i \in K$ be such that $k_i \cdot c_0 = c_i$. The lacunarity of C asserts that the sets $U \cdot c_i = Uk_i \cdot c_0$ are supposed to be pairwise disjoint, which necessitates Uk_i to be pairwise disjoint for $0 \leq i \leq N$. Clearly, $Uk_i \subseteq UK$ since $k_i \in K$. Using the right-invariance of λ, we have $\lambda(UK) \geq \lambda\big(\bigsqcup_{i \leq N} Uk_i\big) = (N+1)\lambda(U)$, and thus $N + 1 \leq \frac{\lambda(UK)}{\lambda(U)}$, as claimed.

We may now use [34, Prop. 4.6] to deduce the existence of a finite partition $C = \bigsqcup_i C_i$ such that no two points in C_i are adjacent. In other words, if $c, c' \in C_i$ are distinct, then $c \notin K \cdot c'$, and therefore $V \cdot c \cap V \cdot c' = \varnothing$, which shows that each C_i is V-lacunary. ∎

Every cross-section C gives rise to a smooth subrelation of $\mathcal{R}_G$ by associating to $x \in X$ "the closest point" of C in the same orbit. Such a subrelation is known as the Voronoi tessellation. For the purposes of Chapter 5, we need a slightly more abstract concept of a tessellation, which may not correspond to Voronoi domains. While far from being the most general, the following treatment is sufficient for our needs.

Definition E.3. Let $G \curvearrowright X$ be a Borel action of a locally compact Polish group on a standard Borel space and let $C \subseteq X$ be a cross-section. A **tessellation** over C is a Borel set $\mathcal{W} \subseteq C \times X$ such that

(1) all fibers $\mathcal{W}_c = \{x \in X : (c, x) \in \mathcal{W}\}$ are pairwise disjoint for $c \in C$;

(2) for all $c \in C$, elements of $\mathcal{W}_c$ are $\mathcal{R}_G$-equivalent to c, i.e., $\{c\} \times \mathcal{W}_c \subseteq \mathcal{R}_G$;

(3) fibers cover the phase space, $X = \bigsqcup_{c \in C} \mathcal{W}_c$.

A tessellation $\mathcal{W}$ over C is called N-**lacunary** for an open subset $N \subseteq G$ if, for every $c \in C$, the inclusion $N \cdot c \subseteq \mathcal{W}_c$ holds. It is said to be K-**cocompact**, where

$K \subseteq G$ is a compact subset, if $\mathcal{W}_c \subseteq K \cdot c$ for all $c \in C$. We say that $\mathcal{W}$ is **cocompact** if it is K-cocompact for some compact subset $K \subseteq G$.

Any tessellation $\mathcal{W}$ can be viewed as a (flipped) graph of a function, since for any $x \in X$, there is a unique $c \in C$ such that $(c, x) \in \mathcal{W}$. We denote such c by $\pi_{\mathcal{W}}(x)$, which produces a Borel map $\pi_{\mathcal{W}} : X \to C$. There is a natural equivalence relation $\mathcal{R}_{\mathcal{W}}$ associated with the tessellation. Namely, x_1 and x_2 are $\mathcal{R}_{\mathcal{W}}$-equivalent whenever they belong to the same fiber, i.e., $\pi_{\mathcal{W}}(x_1) = \pi_{\mathcal{W}}(x_2)$. In view of item (2), $\mathcal{R}_{\mathcal{W}} \subseteq \mathcal{R}_G$, and moreover, every $\mathcal{R}_G$-class consists of countably many $\mathcal{R}_{\mathcal{W}}$-classes.

E.3 Voronoi tessellations

Voronoi tessellations provide a specific method for constructing tessellations over a given cross-section. Suppose that the group G is endowed with a compatible *proper* norm $\|\cdot\|$. Let $D : \mathcal{R}_G \to \mathbb{R}^{\geq 0}$ be the associated metric on the orbits of the action (as in Section 2.1), and let $\preceq_C$ be a Borel linear order on C. The **Voronoi tessellation** over the cross-section C relative to a proper norm $\|\cdot\|$ is the set $\mathcal{V}_C \subseteq C \times X$ defined by

$$\begin{aligned}\mathcal{V}_C = \big\{(c, x) \in C \times X \ : \ & c\mathcal{R}_G x \text{ and for all } c' \in C \text{ such that } c'\mathcal{R}_G x, \text{ either} \\ & D(c, x) < D(c', x) \text{ or} \\ & (D(c, x) = D(c', x) \text{ and } c \preceq_C c')\big\}.\end{aligned}$$

The properness of the norm ensures that for each $x \in X$, there are only finitely many candidates c that minimize $D(c, x)$, and hence each $x \in X$ is associated with a unique $c \in C$. Here is a basic application of Voronoi tessellations.

Proposition E.4. *Let G be a locally compact Polish group acting in a Borel manner on a standard Borel space X. There exists a sequence of cocompact tessellations $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that $\mathcal{R}_G = \bigcup_n \mathcal{R}_{\mathcal{V}_n}$.*

Remark E.5. It is essential that in the statement above, the equivalence relations $\mathcal{R}_{\mathcal{V}_n}$ are *not* required to be nested. For a related statement where these relations are indeed nested and the acting group is amenable, the reader is referred to Lemma 5.1.

Proof of Proposition E.4. Let C be a cocompact cross-section, and let $\|\cdot\|$ be a proper norm on G inducing the metric D on the orbits. For each $n \in \mathbb{N}$, define U_n to be the open ball of radius n centered at 1_G. We denote the Voronoi tessellation over C by $\mathcal{V}_C$.

Lemma E.2 lets us pick a finite sequence of cocompact cross-sections $C_1^n, \ldots, C_{k_n}^n$ such that each C_i^n is U_n-lacunary and $C = \bigsqcup_{i=1}^{k_n} C_i^n$. Notably, the U_n-lacunarity condition ensures that for distinct $c, c' \in C_i^n$, we have $D(c, c') \geq n$. Let $\mathcal{V}_i^n$ denote the Voronoi tessellation over C_i^n.

We claim that the tessellations C_i^n constitute the desired sequence. Let $x, y \in X$ be such that $(x, y) \in \mathcal{R}_G$. Set $c = \pi_{\mathcal{V}_C}(x)$. If n is so large that $D(x, c) < n$ and $D(y, c) < n$, then $\pi_{\mathcal{V}_i^{2n}}(y) = c = \pi_{\mathcal{V}_i^{2n}}(x)$ for the index i satisfying $c \in C_i^{2n}$. We conclude that $(x, y) \in \mathcal{R}_{\mathcal{V}_i^{2n}}$, and the claim follows. ∎

For the sake of Chapter 5, we also need a definition of the Voronoi tessellation for norms that may not be proper. The set $\mathcal{V}_C$ specified as above may, in this case, fail to satisfy item (3) of the definition of a tessellation. For some $x \in X$, there may be infinitely many $c \in C$ that minimize $D(c, x)$, none of which are $\preceq_C$-minimal. We therefore need a different way to resolve the points on the "boundary" between the regions, which can be achieved, for example, by delegating this task to a proper norm.

Definition E.6. Let $\|\cdot\|$ be a compatible norm on G and let C be a cross-section. Pick a compatible proper norm $\|\cdot\|'$ on G and a Borel linear order $\preceq_C$ on C. Let D and D' be the metrics on the orbits of the action associated with the norms $\|\cdot\|$ and $\|\cdot\|'$, respectively. The **Voronoi tessellation** over the cross-section C relative to the norm $\|\cdot\|$ is the set $\mathcal{V}_C \subseteq C \times X$ defined by

$$\begin{aligned}\mathcal{V}_C = \big\{(c, x) \in C \times X \ : \ & c\mathcal{R}_G x \text{ and for all } c' \in C \text{ such that } c'\mathcal{R}_G x \text{ either} \\ & D(c, x) < D(c', x) \text{ or} \\ & (D(c, x) = D(c', x) \text{ and } D'(c, x) < D'(c', x)) \text{ or} \\ & (D(c, x) = D(c', x) \text{ and } D'(c, x) = D'(c', x) \text{ and } c \preceq_C c')\big\}.\end{aligned}$$

The definition of the Voronoi tessellation does depend on the choice of the norm $\|\cdot\|'$ and the linear order $\preceq_C$ on the cross-section, but its key properties remain the same regardless of these choices. We therefore often do not explicitly specify which $\|\cdot\|'$ and $\preceq_C$ are picked. Note also that if the cross-section is cocompact, then every region of the Voronoi tessellation is bounded, i.e., $\sup_{x \in X} D(x, \pi_{\mathcal{V}_C}(x)) < +\infty$.

Our goal is to show that the equivalence relations $\mathcal{R}_{\mathcal{W}}$ are atomless in the sense of Section D, provided that each orbit of the action is uncountable. To this end, we first need the following lemma.

Lemma E.7. *Let G be a locally compact Polish group acting on a standard Lebesgue space (X, μ) by measure-preserving automorphisms. Suppose that almost every orbit of the action is uncountable. If $\mathcal{A} \subseteq X$ is a measurable set such that the intersection of $\mathcal{A}$ with almost every orbit is countable, then $\mu(\mathcal{A}) = 0$.*

Proof. Pick a proper norm $\|\cdot\|$ on G. Let C be a cross-section for the action, and let $B_{2r} \subseteq G$ be an open ball around the identity of sufficiently small radius $2r > 0$ such that $B_{2r} \cdot c \cap B_{2r} \cdot c' = \varnothing$ whenever $c, c' \in C$ are distinct. Let $\mathcal{V}_C$ be the Voronoi tessellation over C relative to $\|\cdot\|$. Note that $B_{2r} \cdot c$ is fully contained in the $\mathcal{R}_{\mathcal{V}_C}$-class of c.

We claim that it is enough to consider the case when $\mathcal{A}$ intersects each $\mathcal{R}_{\mathcal{V}_C}$-class in at most one point. Indeed, the restriction of $\mathcal{R}_{\mathcal{V}_C}$ onto $\mathcal{A}$ is a smooth countable equivalence relation, so one can write $\mathcal{A} = \bigsqcup_{n\in\mathbb{N}} \mathcal{A}'_n$, where each $\mathcal{A}'_n$ intersects each $\mathcal{R}_{\mathcal{V}_C}$-class in at most one point. To simplify notation, we assume that $\mathcal{A}$ already possesses this property.

Let $Y = B_r \cdot C$, and let $(g_n)_{n\in\mathbb{N}}$ be a countable dense subset of G. We define the function $\gamma : X \to \mathbb{N}$ by

$$\gamma(x) = \min\{n \in \mathbb{N} : x \,\mathcal{R}_{\mathcal{V}_C}\, g_n x \text{ and } g_n x \in Y\}.$$

Let $\mathcal{A}_n = \mathcal{A} \cap \gamma^{-1}(n)$, and note that the sets $\mathcal{A}_n$ partition $\mathcal{A}$. It is therefore enough to show that $\mu(\mathcal{A}_n) = 0$ for any $n \in \mathbb{N}$. Pick $n_0 \in \mathbb{N}$. The action is measure-preserving, and therefore $\mu(\mathcal{A}_{n_0}) = \mu(g_{n_0}\mathcal{A}_{n_0})$. Set $\mathcal{B}_0 = g_{n_0}\mathcal{A}_{n_0}$ and note that for any $x \in \mathcal{B}_0$ and $g \in B_r \subseteq G$, one has $gx\mathcal{R}_{\mathcal{V}_C}x$. If the action were free, we could easily conclude that $\mu(\mathcal{B}_0) = 0$, since the sets $g\mathcal{B}_0$, $g \in B_r$, would be pairwise disjoint. In general, we need to exhibit a little more care and construct a countable family of pairwise disjoint sets $\mathcal{B}_n$ as follows. For $x \in \mathcal{B}_0$, let

$$\tau_n(x) = \min\Big\{m \in \mathbb{N} : x\mathcal{R}_{\mathcal{V}_C} g_m x \text{ and } g_m x \notin \bigcup_{k\le n} \mathcal{B}_n\Big\}.$$

The value $\tau_n(x)$ is well-defined because the stabilizer of x is closed and must be nowhere dense in B_r due to the orbit $G \cdot x$ being uncountable. Put $\mathcal{B}_{n+1} = \{g_{\tau_n(x)}x : x \in \mathcal{B}_0\}$ and note that $\mu(\mathcal{B}_n) = \mu(\mathcal{B}_0)$. We get a pairwise disjoint infinite family of sets $\mathcal{B}_n$, all having the same measure. Since μ is finite, we conclude that $\mu(\mathcal{B}_0) = 0$, and the lemma follows. ■

Corollary E.8. *Let G be a locally compact Polish group acting on a standard Lebesgue space (X, μ) by measure-preserving automorphisms. Let C be a cross-section for the action, and let $\mathcal{W} \subseteq C \times X$ be a tessellation. If μ-almost every orbit of G is uncountable, then $\mathcal{R}_{\mathcal{W}}$ is atomless.*

Proof. Consider the disintegration $(\mu_c)_{c\in C}$ of $\mathcal{R}_{\mathcal{W}}$ relative to $(\pi_{\mathcal{W}}, \nu)$, where $\pi_{\mathcal{W}} : X \to C$ and $\nu = (\pi_{\mathcal{W}})_*\mu$. Let $X_a \subseteq X$ be the set of atoms of the disintegration. Since ν-almost every μ_c is finite, the fibers $\pi_{\mathcal{W}}^{-1}(c)$ have countably many atoms. Since every tessellation has only countably many tiles within each orbit, we conclude that X_a has a countable intersection with almost every orbit of the action. Lemma E.7 applies and shows that $\mu(X_a) = 0$. Hence, $\mathcal{R}_{\mathcal{W}}$ is atomless as required. ■

Consider the full group $[\mathcal{R}_{\mathcal{W}}]$, which, by Proposition D.3 and Corollary E.8, is isomorphic to $\mathrm{L}^0(Y, \nu, \mathrm{Aut}([0,1], \lambda))$ for some standard Lebesgue space (Y, ν). This full group can naturally be viewed as a subgroup of $[\mathcal{R}_G]$, and the topology induced on $[\mathcal{R}_{\mathcal{W}}]$ from the full group $[\mathcal{R}_G]$ coincides with the topology of convergence in measure on $\mathrm{L}^0(Y, \nu, \mathrm{Aut}([0,1], \lambda))$ (see Section 3 of [11]). We therefore have the following corollary.

Corollary E.9. *Let G be a locally compact Polish group acting on a standard Lebesgue space (X, μ) by measure-preserving automorphisms. Let C be a cross-section for the action, let $\mathcal{W} \subseteq C \times X$ be a tessellation, and let $\pi_{\mathcal{W}} : X \to C$ be the corresponding reduction. If μ-almost every orbit of G is uncountable, then the subgroup $[\mathcal{R}_{\mathcal{W}}] \leq [\mathcal{R}_G]$ is isomorphic as a topological group to $\mathrm{L}^0(C, (\pi_{\mathcal{W}})_*\mu, \mathrm{Aut}([0,1], \lambda))$. If moreover all orbits of the action have measure zero, then $(\pi_{\mathcal{W}})_*\mu$ is non-atomic, and $[\mathcal{R}_{\mathcal{W}}]$ is isomorphic to $\mathrm{L}^0([0,1], \lambda, \mathrm{Aut}([0,1], \lambda))$.*

Appendix F

Conditional measures

The ergodic decomposition theorem, as formulated in Section E.1, is not available for general probability measure-preserving actions of Polish groups. Conditional measures provide a useful framework to remedy this. As before, $\mathrm{Aut}(X,\mu)$ stands for the group of measure-preserving automorphisms of a standard probability space. It is more useful, however, to view $\mathrm{Aut}(X,\mu)$ as the group of measure-preserving automorphisms of the **measure algebra** $\mathrm{MAlg}(X,\mu)$ of (X,μ), i.e., the Boolean algebra of equivalence classes of Borel subsets of X, identified up to measure zero. The measure algebra is endowed with a natural metric d_μ given by $d_\mu(A,B)=\mu(A\vartriangle B)$. The completeness of $(\mathrm{MAlg}(X,\mu),d_\mu)$ follows directly from its natural isometric identification with $(\mathrm{L}^1(X,\mu,\mathbb{Z}/2\mathbb{Z}),\tilde{d}_Y)$, where $\mathbb{Z}/2\mathbb{Z}$ is endowed with the discrete metric d_Y, and the metric $\tilde{d}_Y$ is given in Definition B.5.

Proposition F.1. *The metric space* $(\mathrm{MAlg}(X,\mu),d_\mu)$ *is complete.* ■

Note that closed subalgebras of $\mathrm{MAlg}(X,\mu)$ are in a one-to-one correspondence with complete (in the measure-theoretical sense) σ-subalgebras of the σ-algebra of Lebesgue measurable sets.

F.1 Conditional expectations

We give a concise overview of how conditional expectations can be defined without the need for disintegration.

Let M be a closed subalgebra of $\mathrm{MAlg}(X,\mu)$ and let $\mathrm{L}^2(M,\mu)$ denote the L^2 space of real-valued M-measurable functions. Note that $\mathrm{L}^2(M,\mu)$ is a closed subspace of $\mathrm{L}^2(X,\mu)=\mathrm{L}^2(\mathrm{MAlg}(X,\mu),\mu)$. The M-**conditional expectation** is the orthogonal projection $\mathbb{E}_M:\mathrm{L}^2(X,\mu)\to\mathrm{L}^2(M,\mu)$. It is uniquely defined by the condition

$$\int_X fg\,d\mu=\int_X \mathbb{E}_M(f)g\,d\mu \quad \text{for all } f\in\mathrm{L}^2(X,\mu) \text{ and all } g\in\mathrm{L}^2(M,\mu). \tag{F.1}$$

Due to the density of step functions in $\mathrm{L}^2(M,\mu)$, the conditional expectation can equivalently be defined as the linear contraction $\mathrm{L}^2(X,\mu)\to\mathrm{L}^2(M,\mu)$ satisfying

$$\int_A f\,d\mu=\int_A \mathbb{E}_M(f)\,d\mu \quad \text{for all } A\in M \text{ and all } f\in\mathrm{L}^2(X,\mu). \tag{F.2}$$

Positive functions are precisely those that yield a non-negative dot product with any characteristic function. By allowing g in Eq. (F.1) to vary over the set of all characteristic functions of subsets in M, we can see that the conditional expectation $\mathbb{E}_M$ preserves positivity.

Proposition F.2. *If $f \in \mathrm{L}^2(X,\mu)$ is non-negative, $f \geq 0$, then $\mathbb{E}_M(f) \geq 0$.*

While we defined conditional expectations as operators on $\mathrm{L}^2(X,\mu)$, their domain can be extended to all of $\mathrm{L}^1(X,\mu)$, making $\mathbb{E}_M$ a contraction from $\mathrm{L}^1(X,\mu)$ to $\mathrm{L}^1(M,\mu)$. This is justified by the following proposition.

Proposition F.3. *The conditional expectation $\mathbb{E}_M : \mathrm{L}^2(X,\mu) \to \mathrm{L}^2(M,\mu)$ is a contraction when the domain and the range are endowed with the L^1 norms.*

Proof. If $f \in \mathrm{L}^2(X,\mu)$ is non-negative, $f \geq 0$, then Eq. (F.1) yields

$$\|f\|_1 = \int_X f\,d\mu = \int_X f \cdot 1\,d\mu = \int_X \mathbb{E}_M(f) \cdot 1\,d\mu = \int_X \mathbb{E}_M(f)\,d\mu.$$

Since $\mathbb{E}_M(f) \geq 0$ by Proposition F.2, we conclude that $\|\mathbb{E}_M(f)\|_1 = \|f\|_1$ for all non-negative $f \in \mathrm{L}^2(X,\mu)$.

An arbitrary $f \in \mathrm{L}^2(X,\mu)$ can be written as the difference $f^+ - f^-$ of non-negative functions $f^+ = \max\{f,0\}$ and $f^- = \max\{-f,0\}$. Note that $f^+, f^- \in \mathrm{L}^2(X,\mu)$ and $\|f^+\|_1 + \|f^-\|_1 = \|f\|_1$. We therefore have

$$\|\mathbb{E}_M(f)\|_1 = \|\mathbb{E}_M(f^+ - f^-)\|_1 \leq \|\mathbb{E}_M(f^+)\|_1 + \|\mathbb{E}_M(f^-)\|_1 = \|f^+\|_1 + \|f^-\|_1 = \|f\|_1,$$

showing that $\mathbb{E}_M$ is a contraction in the L^1 norm. ∎

Remark F.4. By the previous proposition, $\mathbb{E}_M$ admits a (necessarily unique) extension to a contraction

$$\mathbb{E}_M : \mathrm{L}^1(X,\mu) \to \mathrm{L}^1(M,\mu).$$

Moreover, since every non-negative integrable function can be written as an increasing limit of bounded non-negative functions, the analog of Proposition F.2 continues to hold for $f \in \mathrm{L}^1(X,\mu)$.

F.2 Conditional measures

Throughout this section, we let $\chi_A : X \to \{0,1\}$ denote the characteristic function of $A \subseteq X$.

Definition F.5. Let M be a closed subalgebra of $\mathrm{MAlg}(X,\mu)$. The ***M*-conditional measure** of $A \in \mathrm{MAlg}(X,\mu)$, denoted by $\mu_M(A)$, is the conditional expectation of the characteristic function of A, i.e., $\mu_M(A) = \mathbb{E}_M(\chi_A)$.

In particular, the conditional measure $\mu_M(A)$ is an M-measurable function. It enjoys the following natural properties.

Proposition F.6. *Let $M \subseteq \mathrm{MAlg}(X,\mu)$ be a closed subalgebra. The following properties hold for all $A \in \mathrm{MAlg}(X,\mu)$:*

(1) *$\mu_M(\varnothing) = 0$ and $\mu_M(X) = 1$, where 0 and 1 denote the constant maps;*
(2) *$\mu_M(A)$ takes values in $[0,1]$ and $\int_X \mu_M(A) = \mu(A)$;*
(3) *μ_M is σ-additive: if $A = \bigsqcup_n A_n$, $A_n \in \mathrm{MAlg}(X,\mu)$, is a partition, then*

$$\mu_M(A) = \sum_{n\in\mathbb{N}} \mu_M(A_n),$$

where the convergence holds in $\mathrm{L}^1(M,\mu)$;

(4) *if $T \in \mathrm{Aut}(X,\mu)$ fixes every element of M, then $\mu_M(A) = \mu_M(T(A))$.*

Proof. The first item is clear from the fact that both $\varnothing$ and X belong to M, so their characteristic functions are fixed by $\mathbb{E}_M$. The second item follows from the first and positivity of the conditional expectation; the equality is a direct consequence of Eq. (F.2). The third one is a consequence of the L^1 continuity of $\mathbb{E}_M$ and its linearity, noting that $\chi_A = \sum_n \chi_{A_n}$ in $\mathrm{L}^1(M,\mu)$.

Finally, the last item follows from the uniqueness of conditional expectation given by Eq. (F.2). Indeed, if an automorphism T fixes every element of M, then

$$\int_B f\circ T^{-1}\, d\mu = \int_{T(B)} f\, d\mu = \int_B f\, d\mu \qquad \text{for all } B \in \mathrm{MAlg}(X,\mu),$$

so $\mathbb{E}_M(f\circ T^{-1}) = \mathbb{E}_M(f)$. Taking $f = \chi_A$ for $A \in \mathrm{MAlg}(X,\mu)$, we conclude that $\mu_M(T(A)) = \mu_M(A)$. ∎

F.3 Conditional measures and full groups

Conditional measures, as defined in Section F.2, are associated with closed subalgebras of $\mathrm{MAlg}(X,\mu)$. Each subgroup $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ gives rise to the subalgebra of $\mathbb{G}$-invariant sets, and we may therefore associate a conditional measure with the group $\mathbb{G}$ itself.

Definition F.7. Let $\mathbb{G}$ be a subgroup of $\mathrm{Aut}(X,\mu)$. The closed **subalgebra of $\mathbb{G}$-invariant sets** is denoted by $M_{\mathbb{G}}$ and consists of all $A \in \mathrm{MAlg}(X,\mu)$ such that $gA = A$ for all $g \in \mathbb{G}$.

By definition, $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ is **ergodic** if $M_{\mathbb{G}} = \{\varnothing, X\}$. Since $\{\varnothing, X\}$-measurable functions are constants, the $M_{\mathbb{G}}$-conditional measure corresponds to the measure μ when $\mathbb{G}$ is ergodic. The following lemma is an easy consequence of the definitions of the full group generated by a subgroup (Section 3.1) and the weak topology on $\mathrm{Aut}(X,\mu)$.

Lemma F.8. *Let $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ be a group.*

(1) *If $[\mathbb{G}]$ is the full group generated by $\mathbb{G}$, then $M_{\mathbb{G}} = M_{[\mathbb{G}]}$.*

(2) *If $\Gamma \leq \mathbb{G}$ is dense in the weak topology, then $M_\Gamma = M_{\mathbb{G}}$.*

Given a subgroup $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$, we denote the $M_{\mathbb{G}}$-conditional measure simply by $\mu_{\mathbb{G}}$.

Recall that a **partial measure-preserving automorphism** of (X, μ) is a measure-preserving bijection $\varphi : \operatorname{dom}\varphi \longrightarrow \operatorname{rng}\varphi$ between measurable subsets of X, called the domain and the range of φ, respectively. The **pseudo full group** generated by a group $\Gamma \leq \mathrm{Aut}(X, \mu)$ is denoted by $[\![\Gamma]\!]$ and consists of all partial automorphisms $\varphi : \operatorname{dom}\varphi \longrightarrow \operatorname{rng}\varphi$ for which there exists a partition $\operatorname{dom}\varphi = \bigsqcup_n A_n$ and elements $\gamma_n \in \Gamma$ such that $\varphi \restriction_{A_n} = \gamma_n \restriction_{A_n}$ for all n. Elements of $[\![\Gamma]\!]$ automatically preserve the conditional measure μ_Γ in the sense that if $A \subseteq \operatorname{dom}\varphi$, then $\mu_\Gamma(\varphi(A)) = \mu_\Gamma(A)$. Indeed,

$$
\begin{aligned}
\mu_\Gamma(\varphi(A)) &= \mu_\Gamma\Big(\varphi\Big(\bigsqcup_n (A \cap A_n)\Big)\Big) = \mu_\Gamma\Big(\bigsqcup_n \varphi(A \cap A_n)\Big) \\
\because \text{Prop. F.6(3)} &= \sum_n \mu_\Gamma(\varphi(A \cap A_n)) = \sum_n \mu_\Gamma(\gamma_n(A \cap A_n)) \\
\because \text{Prop. F.6(4)} &= \sum_n \mu_\Gamma(A \cap A_n) = \mu_\Gamma(A).
\end{aligned}
$$

Lemma F.9. *Let $\mathbb{G} \leq \mathrm{Aut}(X, \mu)$ be a group. For all $A, B \in \mathrm{MAlg}(X, \mu)$ satisfying $\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(B)$, there exists an element $\varphi \in [\![\mathbb{G}]\!]$ such that* $\operatorname{dom}\varphi = A$ *and* $\operatorname{rng}\varphi = B$.

Proof. Let $\Gamma = \{\gamma_n : n \in \mathbb{N}\}$ be a countable weakly dense subgroup of $\mathbb{G}$. It follows from Lemma F.8 that $\mu_\Gamma(A) = \mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(B) = \mu_\Gamma(B)$, and it is evident that $[\![\Gamma]\!] \leq [\![\mathbb{G}]\!]$.

We inductively define sequences $(A_n)_n$ and $(B_n)_n$ of subsets of A and B, respectively, starting with $A_0 = A \cap \gamma_0^{-1} B$ and $B_0 = \gamma_0 A_0$, and for $n \geq 1$, we set

$$
A_n = \Big(A \setminus \bigcup_{m<n} A_m\Big) \cap \gamma_n^{-1}\Big(B \setminus \bigcup_{m<n} B_m\Big) \quad \text{and} \quad B_n = \gamma_n A_n.
$$

By construction, the sets A_n are pairwise disjoint subsets of A, each set satisfies $\gamma_n A_n = B_n$, and the sets B_n are pairwise disjoint subsets of B. We assert that $\varphi = \bigsqcup_n (\gamma_n \restriction_{A_n})$ is the desired element of $[\![\mathbb{G}]\!]$.

Suppose, towards a contradiction, that either $\operatorname{dom}\varphi \neq A$ or $\operatorname{rng}\varphi \neq B$. Since Γ preserves μ_Γ and $\mu_\Gamma(A) = \mu_\Gamma(B)$, the sets $A \setminus \operatorname{dom}\varphi$ and $B \setminus \operatorname{rng}\varphi$ have the same M_Γ-conditional measure, which is not constantly equal to zero. The set

$$
\tilde{A} = \bigcup_{\gamma \in \Gamma} \gamma(A \setminus \operatorname{dom}\varphi)
$$

is Γ-invariant and non-zero. Its conditional measure is therefore the characteristic function $\chi_{\tilde{A}}$, which must be greater than or equal to $\mu_\Gamma(A \setminus \operatorname{dom}\varphi) = \mu_\Gamma(B \setminus \operatorname{rng}\varphi)$.

We conclude that $B \setminus \operatorname{rng}\varphi \subseteq \bigcup_{\gamma\in\Gamma}\gamma(A \setminus \operatorname{dom}\varphi)$. In particular, there is the first $n \in \mathbb{N}$ such that $(A \setminus \operatorname{dom}\varphi) \cap \gamma_n^{-1}(B \setminus \operatorname{rng}\varphi)$ is non-zero. By construction, this set should be a subset of A_n, yielding the desired contradiction. ■

Proposition F.10. *Let $\mathbb{G}$ be a full subgroup of* $\operatorname{Aut}(X,\mu)$. *The following conditions are equivalent for all $A, B \in \operatorname{MAlg}(X,\mu)$:*

(1) *$\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(B)$;*

(2) *there is $T \in \mathbb{G}$ such that $T(A) = B$;*

(3) *there is an involution $T \in \mathbb{G}$ such that $T(A) = B$ and $\operatorname{supp} T = A \triangle B$.*

Proof. The implication (2)⇒(1) is a direct consequence of the definition of $M_{\mathbb{G}}$ along with item (4) of Proposition F.6. Also, (3)⇒(2) is evident.

We now prove the implication (1)⇒(3). The assumption $\mu_{\mathbb{G}}(A) = \mu_{\mathbb{G}}(A)$ guarantees that $\mu_{\mathbb{G}}(A \setminus B) = \mu_{\mathbb{G}}(B \setminus A)$. Lemma F.9 applies and produces an element $\varphi \in [\![\mathbb{G}]\!]$ such that $\varphi(A \setminus B) = B \setminus A$. The involution $\varphi \sqcup \varphi^{-1} \sqcup \mathrm{id}_{X\setminus(A\triangle B)}$ meets the required conditions. ■

F.4 Aperiodicity

A countable subgroup $\Gamma \leq \operatorname{Aut}(X,\mu)$ is called **aperiodic** if almost all the orbits of some (equivalently, any) realization of its action on (X,μ) are infinite. The so-called Maharam's lemma provides a characterization of aperiodicity in a purely measure-algebraic way. We begin by formulating a variant of the standard marker lemma for countable Borel equivalence relations (see, for instance, [33, Lemma 6.7]).

Lemma F.11. *Let $\Gamma \curvearrowright X$ be a Borel action of a countable group on a standard Borel space X. For every Borel set $C \subseteq X$, there is a decreasing sequence $(C_n)_n$ of Borel subsets of C such that $C \subseteq \Gamma \cdot C_n$ for each n, and the set $\bigcap_n C_n$ intersects the Γ-orbit of every $x \in X$ in at most one point. Furthermore, if all orbits of Γ are infinite, the sets C_n can be chosen to have an empty intersection, $\bigcap_n C_n = \varnothing$.*

The following result is essentially due to H. Dye [15], where it is called Maharam's lemma.

Theorem F.12 (Maharam's lemma). *Let $\Gamma \leq \operatorname{Aut}(X,\mu)$ be a countable subgroup. The following are equivalent:*

(1) *Γ is aperiodic;*

(2) *for any $A \in \operatorname{MAlg}(X,\mu)$ and any M_Γ-measurable function $f : X \to [0,1]$ satisfying $f \leq \mu_\Gamma(A)$, there is $B \subseteq A$, $B \in \operatorname{MAlg}(X,\mu)$, such that $\mu_\Gamma(B) = f$.*

Proof. Let us start with the simpler implication (2)⇒(1), which is proved using a contrapositive argument. Assume that (1) fails, meaning that Γ is not aperiodic. Let

$n \in \mathbb{N}$ be an integer such that the Γ-invariant set $X_n = \{x \in X : |\Gamma \cdot x| = n\}$ has non-zero measure. We may assume that X bears a Borel total order (for instance, by identifying X with $[0,1]$). Let $A = \{x \in X_n : x = \max\{\Gamma \cdot x\}\}$ be the set of maximal points of the n-element Γ-orbits and set φ, $\operatorname{dom} \varphi = X_n \setminus A$, to be the element of the pseudo full group $[\![\Gamma]\!]$ that takes every $x \in X_n \setminus A$ to its $<$-successor in the orbit $\Gamma \cdot x$. Given any $B \subseteq A$, the set $\bigsqcup_{k=0}^{n-1} \varphi^{-k}(B)$ is Γ-invariant, hence $\mu_\Gamma(\bigsqcup_{k=0}^{n-1} \varphi^{-k}(B))$ takes values in $\{0,1\}$. Also

$$\mu_\Gamma\Big(\bigsqcup_{k=0}^{n-1} \varphi^{-k}(B)\Big) = \sum_{k=0}^{n-1} \mu_\Gamma\big(\varphi^{-k}(B)\big) = n\mu_\Gamma(B),$$

where the last equality is a consequence of Proposition F.6. We conclude that $\mu_\Gamma(B)$ necessarily takes values in $\{0, \frac{1}{n}\}$, which contradicts (2).

We now assume that Γ is aperiodic and prove the direct implication (1)$\Rightarrow$(2). The argument is based on the following claim.

Claim. For every $C \in \mathrm{MAlg}(X,\mu)$, for every M_Γ-measurable not almost surely zero $f : X \to [0,1]$ such that $f \leq \mu_\Gamma(C)$, there is a non-empty $B \subseteq C$ satisfying $\mu_\Gamma(B) \leq f$.

Proof of the claim. Let $(C_n)_n$ be a sequence of subsets of C given by Lemma F.11. Note that $\mu_\Gamma(C_n) \to 0$ in L^1, since $\bigcap_n C_n = \varnothing$ and the C_n's are decreasing. Passing to a subsequence, we may assume that the convergence $\mu_\Gamma(C_n) \to 0$ holds pointwise. Set $B_n = \{x \in C_n : \mu_\Gamma(C_n)(x) \leq f(x)\}$ and note that $\mu_\Gamma(B_n) \leq \mu_\Gamma(C_n)$ and therefore $\mu_\Gamma(B_n) \leq f$.

Pointwise convergence $\mu_\Gamma(C_n) \to 0$ guarantees the existence of an index n such that $\mu(B_n) > 0$, and so the set $B = B_n$ is as required. $\square_{\text{claim}}$

The conclusion of the theorem now follows from a standard application of Zorn's lemma[1]. The latter provides a maximal family $(B_i)_{i \in I}$ of pairwise disjoint positive measure elements of $\mathrm{MAlg}(X,\mu)$ contained in A and satisfying $\sum_{i \in I} \mu_\Gamma(B_i) \leq f$. The index set I has to be countable, and if $B = \bigsqcup_{i \in I} B_i$ then $\mu_\Gamma(B) = \sum_{i \in I} \mu_\Gamma(B_i) \leq f$. Assume towards a contradiction that $\mu_\Gamma(B)$ is not equal to f almost everywhere, and use the previous claim to get a non null $B' \subseteq A \setminus B$ with $\mu_\Gamma(B') \leq f - \mu_\Gamma(B)$, contradicting the maximality of $(B_i)_{i \in I}$. Therefore, $\mu_\Gamma(B) = f$ as claimed. ∎

We conclude this appendix with a useful consequence of aperiodicity. Recall that in Definition 3.6, we define a potentially uncountable subgroup $G \leq \mathrm{Aut}(X,\mu)$ to be aperiodic if it contains an aperiodic countable subgroup. Furthermore, this implies that G contains a countable weakly dense aperiodic subgroup.

Lemma F.13. *Let $\mathbb{G} \leq \mathrm{Aut}(X,\mu)$ be an aperiodic full group. For each set $B \in \mathrm{MAlg}(X,\mu)$, there is an involution $U \in \mathbb{G}$ whose support is equal to B.*

[1] A more constructive version of the whole argument can be found in [38, Prop. D.1].

Proof. Let $\Gamma \leq \mathbb{G}$ be a countable weakly dense aperiodic subgroup of $\mathbb{G}$. Then, by weak density, $M_\Gamma = M_\mathbb{G}$ and $\mu_\Gamma = \mu_\mathbb{G}$. Theorem F.12 thus provides $A \subseteq B$ such that $\mu_\mathbb{G}(A) = \mu_\mathbb{G}(B)/2$. We then have

$$\mu_\mathbb{G}(B \setminus A) = \mu_\mathbb{G}(B) - \mu_\mathbb{G}(B)/2 = \mu_\mathbb{G}(A),$$

and item (3) of Proposition F.10 provides an involution $T \in \mathbb{G}$ satisfying $T(B \setminus A) = A$ and $\operatorname{supp} T = (B \setminus A) \vartriangle A = B$. ∎

Remark F.14. Lemma F.13 in fact characterizes the aperiodicity of full groups. If $\mathbb{G}$ is not aperiodic, then there is some $B \in \mathrm{MAlg}(X, \mu)$ that is not the support of any involution. This is because its $M_\mathbb{G}$-conditional measure cannot be split in half, as shown in the proof of the direct implication in Theorem F.12.

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